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# GA—68—2023

**FACULTY OF ARTS/SCIENCE**

**B.Sc. (Fifth Semester) EXAMINATION**

**MARCH/APRIL, 2023**

**(CBCS/New Pattern)**

MATHEMATICS

Paper XII

(Metric Spaces)

**(Wednesday, 3-5-2023)** **Time : 10.00 a.m. to 12.00 noon**

*Time— Two Hours* *Maximum Marks*—40

*N.B.* :— (*i*) Attempt *all* questions.

(*ii*) Figures to the right indicate full marks.

1. Prove that continuous image of compact set is compact. 15

*Or*

(*a*) Let (X, $d$) be any metric space. Prove that a subset F, of X, is closed if and only if its complement, in X, is open. 8

(*b*) Show that the function :

$d : \mathrm{R} \times \mathrm{R} \to \mathrm{R}$ defined by $d(x, y) = |x - y|$, $\forall\ x, y \in \mathrm{R}$

is a metric on the set R of all real numbers. 7

2. Let Y be a subset of a metric space (X, $d$), then prove that the following are equivalent : 15

(*i*) Y is connected

(*ii*) Y cannot be expressed as disjoint union of two non-empty closed sets in Y.

P.T.O.

*Or*

(*a*) Let (X, $d_1$) and (Y, $d_2$) be two metric spaces, then prove that $f : X \to Y$ is continuous if and only if $f^{-1}(G)$ is open in X, whenever G is open in Y. 7

(*b*) Prove that every convergent sequence is a Cauchy sequence. 8

3. Attempt any *two* of the following : 5 each

(*a*) Let (X, *d*) be any metric space. Show that the function $d_1$ defined by :

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \text{ for all } x, y \in X$$

is a metric on X.

(*b*) Prove that every compact subset A of a metric space (X, *d*) is bounded.

(*c*) Show that the space G[0, 1] of all bounded continuous real-valued functions defined on the closed interval [0, 1] with the metric d given by :

$$d(f, g) = \max |f(x) - g(x)|$$

is a complete metric space.

(*d*) Discuss the connectedness of a subset :

$$D = \left\{(x, y) \mid x \neq 0, y = \sin \frac{1}{x}\right\}$$

of the Euclidean space $R^2$.