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Research Article

Existence of Solution to a Quadratic Functional Integro-Differential Fractional Equation

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Abstract. An algebraic fixed point theorem involving the two operators in a Banach algebra is used to prove the existence of solutions to fractional order quadratic functional integro-differential equation in $\mathcal{R}_+$. Also, we establish the locally attractivity results and extremal solutions along with suitable example.

Keywords. Banach algebras; Integro-Differential equation; Existence result; Locally attractive solution; Extremal solution

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1. Introduction

Fractional differential equations appear in various fields of engineering and science such as viscoelasticity, electrochemistry, control, electromagnetic, porous media, etc. For example, in [29–32, 36, 40, 41], we can see applications of fractional differential equations in signal processing, complex dynamics in biological tissues, viscoelastic materials, thermal systems and heat conduction. One can see the application of fractional differential equations in complex physical systems, physical systems description and control ([6, 7, 41]).

Nonlinear quadratic integral equations appear very often, in many applications of real world problem. For examples, quadratic integral equations are often applicable in the theory of radiactive transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. Numerous research papers and monographs devoted to quadratic differential and integral equations of fractional order have appeared (see [5,8,11,13–16,20,21,23,24,27, 33–35,39]). These papers contain various types of existence results for equations of fractional order. In this paper, we study the existence results for fractional order quadratic functional Integro-Differential equation .Along with the locally attractivity and extremal solutions along with suitable example.

2. Statement of the Problem

Let $\zeta \in (0,1)$. $\mathcal{R}$ denote the real numbers whereas $\mathcal{R}_+$ be the set of nonnegative numbers i.e. $\mathcal{R}_+ = [0,\infty) \subset \mathcal{R}$.

Consider the functional integro-differential equations of fractional order

$$\frac{d^\zeta}{dt^\zeta}[x(t) - f(t,x(t))] = g\left(t, \int_0^t h(s,x_s)\right) ds \quad \forall\ t \in \mathcal{R}_+, \tag{2.1}$$

where $x_t : \mathcal{R}_+ \to \mathcal{R}$, $f(t,x) = f : \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$, $g(t,x) = g : \mathcal{R}_+ \times \mathcal{R} \times \mathcal{R}$ and by a solution of the (2.1) we mean a function $x \in B\mathcal{C}(\mathcal{R}_+,\mathcal{R})$ that satisfies (2.1) on $\mathcal{R}_+$, and $B\mathcal{C}(\mathcal{R}_+,\mathcal{R})$ is the space of continuous and bounded real-valued functions defined on $\mathcal{R}_+$.

Applying a Krisonoselkii's fixed point theorem [12,25,26] the existence results for QFIDE (2.1) will be obtained.

We collect some preliminary definitions and auxiliary results that will be used in the follows.

3. Preliminaries

Let $X = B\mathcal{C}(\mathcal{R}_+,\mathcal{R})$ be Banach algebra with norm $\|\cdot\|$ and let O be a subset of X. Let a mapping $\mathcal{A} : X \to X$ be an operator and consider the following operator equation in X, namely,

$$x(t) = (\mathcal{A}x)(t) \in \mathcal{R}_+. \tag{3.1}$$

Definition 3.1 ([4])**.** The solution $x(t)$ of the equation (3.1) is said to be locally attractive if there exists an closed ball $B_r[0]$ in $B\mathcal{C}(\mathcal{R}_+,\mathcal{R})$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (3.1) belonging to $B_r[0] \cap \Omega$ such that

$$\lim_{t\to\infty}(x(t) - y(t)) = 0. \tag{3.2}$$

Definition 3.2 ([4])**.** Let X be a Banach space. A mapping $\mathcal{A} : X \to X$ is called Lipschitz if there is a constant $\alpha > 0$ such that $\|\mathcal{A}x - \mathcal{A}y\| = \alpha\|x - y\|$ for all $x,y \in X$ if $\alpha < 1$ then $\mathcal{A}$ is called a contraction on X with the contraction constant α.

Definition 3.3 (Dugundji and Granas [16])**.** An operator $\mathcal{A}$ on a Banach space X into itself is called *compact* if for any bounded subset S of X, $\mathcal{A}(S)$ is a relatively compact subset of X. If $\mathcal{A}$ is continuous and compact, then it is called completely continuous on X.

Let X be a Banach space with the norm $\|\cdot\|$ and let $\mathcal{A}: X \to X$ be an operator (in general nonlinear). Then $\mathcal{A}$ is called

(i) *compact* if $\mathcal{A}(X)$ is relatively compact subset of X;

(ii) *totally bounded* if $\mathcal{A}(S)$ is a totally bounded subset of X for any bounded subset S of X;

(iii) *completely continuous* if it is continuous and totally bounded operator on X.

It is clear that every compact operator is totally bounded but the converse need not be true.

The solutions of (2.1) in the space $B\mathcal{C}(\mathcal{R}_+, \mathcal{R})$ of continuous and bounded real-valued functions defined on $\mathcal{R}_+$. Define a standard supremum norm $\|\cdot\|$ and a multiplication "·" in $B\mathcal{C}(\mathcal{R}_+, \mathcal{R})$ by

$$\|x\| = \sup\{|x(t)| : t \in \mathcal{R}_+\}, \tag{3.3}$$

$$(xy)(t) = x(t)y(t), \quad t \in \mathcal{R}_+. \tag{3.4}$$

Clearly, $B\mathcal{C}(\mathcal{R}_+, \mathcal{R})$ becomes a Banach space with respect to the above norm and the multiplication in it. By $\mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ we denote the space of Lebesgue integrable functions on $\mathcal{R}_+$ with the norm $\|\cdot\|_{\mathcal{L}^1}$ defined by

$$\|x\|_{\mathcal{L}^1} = \int_0^\infty |x(t)|\,dt. \tag{3.5}$$

Denote by $\mathcal{L}^1(a,b)$ be the space of Lebesgue integrable functions on the interval (a,b), which is equipped with the standard norm. Let $x \in \mathcal{L}^1(a,b)$ and let $\beta > 0$ be a fixed number.

Definition 3.4 ([35])**.** The Riemann-Liouville fractional integral of order β of the function $x(t)$ is defined by the formula:

$$I^\beta x(t) = \frac{1}{G(\beta)} \int_0^t \frac{x(s)}{(t-s)^{1-\beta}}\,ds, \quad t \in (a,b), \tag{3.6}$$

where $G(\beta)$ denote the gamma function.

It may be shown that the fractional integral operator I^β transforms the space $\mathcal{L}^1(a,b)$ into itself and has some other properties (see [4, 9, 10, 16, 28, 35, 39]).

Definition 3.5. A set $A \subseteq [a,b]$ is said to be measurable if $m^*A = m_*A$. In this case we define mA, the measure of A as $mA = m^*A = m_*A$.

If A_1 and A_2 are measurable subsets of $[a,b]$ then their union and their intersection is also measurable.

Clearly, every open or closed set in R is measurable.

Definition 3.6. Let f be a function defined on $[a,b]$. Then f is measurable function if for each $\alpha \in R$, the set $\{x : f(x) > \alpha\}$ is measurable set.

i.e. f is measurable function if for every real number α the inverse image of (α, ∞) is an open set.

As (α, ∞) is an open set and if f is continuous, then inverse image under f of (α, ∞) is open sets being measurable, hence every continuous function is measurable.

Definition 3.7. A sequence of functions f_n is said to converge uniformly on an interval $[a,b]$ to a function f if for any $\epsilon > 0$ and for all $x \in [a,b]$ there exists an integer N (dependent only on ϵ) such that for all $x \in [a,b]$

$$|f_n(x) - f(x)| < \epsilon, \quad \forall\, n \geq N.$$

Definition 3.8. The family F is equicontinuous at a point $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ a such that $d(f(x_0), f(x)) < \varepsilon$ for all $f \in F$ and all x that $d(x_0, x) < \delta$.
The family is pointwise equicontinuous if it is equicontinuous at each point of X.
The family is uniformly equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ a such that $d(f(x_1), f(x_2)) < \varepsilon$ for all $f \in F$ and all $x_1, x_2 \in X$ such that $d(x_1, x_2) < \delta$.

Theorem 3.1 (Arzela-Ascoli Theorem [21]). *Every uniformly bounded and equi-continuous sequence $\{f_n\}$ of functions in $\mathcal{C}(J,\mathcal{R})$, has a convergent subsequence.*

Theorem 3.2 ([21]). *A metric space X is compact iff every sequence in X has a convergent subsequence.*

We employ a hybrid fixed point theorem of *Krasnoselskii's* for proving the existence result.

Theorem 3.3 (Krasnoselskii's [12, 25, 26]). *Let X be a Banach space and D be a bounded closed convex subset of X. Let $\mathcal{A}$, $\mathcal{B}$ maps D into X s.t. $\mathcal{A}u + \mathcal{B}u \in D$ for every $(u,v) \in D$.*
If $\mathcal{A}$ is a contraction and $\mathcal{B}$ is completely continuous then the equation $\mathcal{A}w + \mathcal{B}w = w$ has a solution w on D. i.e.

(a) *$\mathcal{A}$ is a contraction,*

(b) *$\mathcal{B}$ is completely continuous,*

(c) *$\mathcal{A}u + Bu \in D$.*

4. Existence Results

Definition 4.1 ([21]). A mapping $g : \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is said to be Caratheodory if

(1) $t \to g(t,x)$ is measurable for all $x \in \mathcal{R}$, and

(2) $x \to g(t,x)$ is continuous almost everywhere for $t \in \mathcal{R}_+$.

Again a caratheodory function g is called $\mathcal{L}^1$-Caratheodory if

(3) for each real number $r > 0$ there exists a function $h_r \in \mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ such that $|g(t,x)| = h_r(t)$ a.e. $t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$ with $|x| = r$.

Finally, a Caratheodory function $g(t,x)$ is called $\mathcal{L}^1_{\mathcal{R}}$-Caratheodory if

(4) there exist a function $h \in \mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ such that $|g(t,x)| = h(t)$.

a.e. $t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$.

For convenience, the function h is referred to as a bound function of g.

We consider the nonlinear quadratic functional integro-differential equation (2.1) under the following assumptions:

(H_1) The function $f(t,x) : \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is continuous and bounded with bound $F = \sup_{(t,x)\in\mathcal{R}_+\to\mathcal{R}} |f(t,x)|$ there exists a bounded function $l : \mathcal{R}_+ \to \mathcal{R}_+$ with bound L satisfying

$$|f(t,x)-f(t,y)| = \frac{l(t)|x-y|}{2(N+|x-y|)}, \quad t \in \mathcal{R}_+, \text{ for all } x,y \in \mathcal{R} \text{ and } 0 < L \leq N$$

and vanishes as $\lim_{t\to\infty}$.

(H_2) The functions $g : \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ satisfy caratheodory condition (i.e. measurable in t for all $x \in \mathcal{R}$ and continuous in x for all $t \in \mathcal{R}_+$) and there exist function $h_1 \in \mathcal{L}^1(\mathcal{R}_+,\mathcal{R})$ such that $g(t,x) \leq h_1(t)\ \forall\ (t,x) \in \mathcal{R}_+ \times \mathcal{R}$.

(H_3) The uniform continuous function $v : \mathcal{R}_+ \to \mathcal{R}_+$ defined by the formulas $v_1(t) = \int_0^t \frac{h_1(s)}{(t-s)^{1-\zeta}} ds$, is bounded on $\mathcal{R}_+$ and vanish at infinity, that is, $\lim_{t\to\infty} v(t) = 0$.

Remark 4.1. Note that if the hypothesis (H_2) hold, then there exist constants $K_1 > 0$ and such that: $K_1 = \sup_{t\geq 0} \frac{1}{G(\zeta)} \int_0^t \frac{h_1(s)}{(t-s)^{1-\zeta}} ds$.

Theorem 4.1. *Suppose that the hypotheses [(H_1)-(H_3)] are hold. Then the equation* (2.1) *has a solution in the space $B\mathcal{C}(\mathcal{R}_+,\mathcal{R})$. Moreover, solutions of the equation* (2.1) *are locally attractive on$\mathcal{R}_+$.*

Proof. By a solution of the (2.1) we mean a continuous function $x : \mathcal{R}_+ \to \mathcal{R}$ that satisfies (2.1) on $\mathcal{R}_+$.

Let $X = B\mathcal{C}(\mathcal{R}_+,\mathcal{R})$ be Banach Algebras of all continuous and bounded real valued function on $\mathcal{R}_+$ with theorem

$$\|x\| = \sup_{t\in\mathcal{R}_+} |x(t)|. \tag{4.1}$$

We show that existence of solution for (2.1) under some suitable conditions on the functions involved in (2.1).

Consider the closed ball $B_r[0]$ in X centered at origin 0 and of radius r, where

$$r = F + (K_1) > 0,$$

$$\frac{d^\zeta}{dt^\zeta}[x(t) - f(t,x(t))] = g\left(t, \int_0^t h(s,x_s)\right) ds,$$

$$x(t) = f(t,x(t)) + I^\zeta g\left(t, \int_0^t h(s,x_s)\right) ds, \quad \forall\ t \in \mathcal{R}_+.$$

Let us define two operators $\mathcal{A}$ and $\mathcal{B}$ on $B_r[0]$ by

$$\mathcal{A}x(t) = f(t,x(t)) \tag{4.2}$$

and

$$\mathcal{B}x(t) = I^\zeta g\left(t, \int_0^t h(s,x_s)\right) ds, \quad \forall\ t \in \mathcal{R}_+. \tag{4.3}$$

In the view of hypotheses (H_1), the mapping $\mathcal{A}$ is well defined and the function $\mathcal{A}x$ is continuous and bounded on $\mathcal{R}_+$. The function $\mathcal{B}x$ is also continuous and bounded in view of hypotheses (H_2).

Therefore, $\mathcal{A}$ and $\mathcal{B}$ define the operators $\mathcal{A},\mathcal{B}: B_r[0] \to X$.
We wish to show that $\mathcal{A}$ and$\mathcal{B}$ satisfy all the requirements of theorem (3.3) on $B_r[0]$.

Step I: Firstly, we show that $\mathcal{A}$ is a contraction mapping. Let $x, y \in X$ be arbitrary, and then by hypothesis (H_1), we get

$$\begin{aligned} |\mathcal{A}x(t) - \mathcal{A}y(t)| &= |f(t,x(t)) - f(t,y(t))| \\ &= \frac{l(t)|x(t)-y(t)|}{2(N+|x-y|)} \\ &= \frac{L|x(t)-y(t)|}{2(N+|x-y|)}, \quad \text{for all } t \in \mathcal{R}_+. \end{aligned} \tag{4.4}$$

Taking supremum over t

$$\|\mathcal{A}x - \mathcal{A}y\| = \frac{L\|x-y\|}{2(N+\|x-y\|)}, \quad \text{for all } x, y \in X. \tag{4.5}$$

This shows that $\mathcal{A}$ is Contraction mapping on X with the contraction constant $L_1 = \frac{L}{2(N+\|x-y\|)}$.

Step II: Secondly, we show that $\mathcal{B}$ is completely continuous operator on $B_r[0]$.

Firstly, we show that $\mathcal{B}$ is continuous on $B_r[0]$.

Case I: Suppose that$t = T$ there exist $T > 0$ and let us fix arbitrary $\varepsilon > 0$ and take $x, y \in B_r[0]$ such that $\|x - y\| = \varepsilon$. Then

$$\begin{aligned} |(\mathcal{B}x)t - (\mathcal{B}y)t| &= \left| I^\zeta g\left(t, \int_0^t h(s,x_s)\right) ds - I^\zeta g\left(t, \int_0^t h(s,y_s)\right) ds \right| \\ &= \left| \frac{1}{G(\zeta)} \int_0^t \frac{g(t, \int_0^t h(s,x_s))ds}{(t-s)^{1-\zeta}} ds - \frac{1}{G(\zeta)} \int_0^t \frac{g(t, \int_0^t h(s,y_s))ds}{(t-s)^{1-\zeta}} ds \right| \\ &= \frac{1}{G(\zeta)} \left[\int_0^t \frac{|g(t, \int_0^t h(s,x_s))ds|}{(t-s)^{1-\zeta}} ds + \int_0^t \frac{|g(t, \int_0^t h(s,y_s))ds|}{(t-s)^{1-\zeta}} ds \right] \\ &= \frac{1}{G(\zeta)} \left[\int_0^t \frac{h_1(s)}{(t-s)^{1-\zeta}} ds + \int_0^t \frac{h_1(s)}{(t-s)^{1-\zeta}} ds \right] \\ &= \frac{2}{G(\zeta)} \left[\int_0^t \frac{h_1(s)}{(t-s)^{1-\zeta}} ds \right] \\ &= \frac{2v(t)}{G(\zeta)}. \end{aligned} \tag{4.6}$$

Hence, we see that there exists $T > 0$, such that $v(t) = \frac{\varepsilon G(\zeta)}{2}$ for $t > T$. Since ε is an arbitrary, from (4.6) we derive that

$$|(\mathcal{B}x)t - (\mathcal{B}y)t| = \varepsilon. \tag{4.7}$$

Case II: Further, let us assume that, $t \in [0,T]$ then evaluating similarly to above we obtain the following estimate

$$|(\mathcal{B}x)t - (\mathcal{B}y)t| = \left| I^\zeta g(t, \int_0^t h(s,x_s))ds - I^\zeta g(t, \int_0^t h(s,y_s))ds \right|$$

$$
\begin{aligned}
&= \left| \frac{1}{G(\zeta)} \int_0^T \frac{g(t, \int_0^t h(s,x_s))}{(t-s)^{1-\zeta}} ds - \frac{1}{G(\zeta)} \int_0^T \frac{g(t, \int_0^t h(s,y_s))}{(t-s)^{1-\zeta}} ds \right| \\
&= \frac{1}{G(\zeta)} \left[\int_0^T \frac{|g(t, \int_0^t h(s,x_s)) - g(t, \int_0^t h(s,y_s))|}{(t-s)^{1-\zeta}} ds \right] \\
&= \frac{1}{G(\zeta)} \left[\int_0^T \frac{w_r^T(g,\epsilon)}{(t-s)^{\zeta}} ds \right] \\
&= \left[\frac{w_r^T(g,\epsilon)}{G(\zeta)\zeta} T^{\zeta} ds \right] \\
&= \left[\frac{w_r^T(g,\epsilon)}{G(\zeta+1)} T^{\zeta} ds \right]
\end{aligned} \tag{4.8}
$$

where

$$
w_r^T(g,\epsilon) = \sup\left\{ \left| g\left(s, \int_0^t h(s,x_s)\right) - g\left(s, \int_0^t h(s,y_s)\right) \right| : s \in [0,T]; x,y \in [-r,r], |x-y| = \epsilon \right\}.
$$

Therefore, from the uniform continuity of the function $g(t,x)$ on the set $[0,T]\times[-r,r]$. We derive that $w_r^T(g,\epsilon) \to 0$ as $\varepsilon \to 0$.

Now, combining the Case I and II, we conclude that the operator $\mathcal{B}$ is continuous operator on closed ball $B_r[0]$ in to itself.

Step III: Next we show that $\mathcal{B}$ is compact on $B_r[0]$.

(A) First prove that every sequence $\{\mathcal{B}x_n\}$ in $\mathcal{B}(B_r[0])$ has a uniformly bounded sequence in $\mathcal{B}(B_r[0])$. Now by (H_1)–(H_3)

$$
\begin{aligned}
|(\mathcal{B}x_n)t| &= \left| \frac{1}{G(\zeta)} \int_0^t \frac{g(t, \int_0^t h(s,{x_s}^n))}{(t-s)^{1-\alpha}} ds \right| \\
&= \frac{1}{G(\zeta)} \int_0^t \frac{|g(t, \int_0^t h(s,{x_s}^n))|}{(t-s)^{1-\zeta}} ds \\
&= \frac{1}{G(\zeta)} \int_0^t \frac{h_1(s)}{(t-s)^{1-\zeta}} ds \\
&= \frac{v(t)}{G(\zeta)} \\
&= K_1, \quad \text{for all } t \in \mathcal{R}_+ .
\end{aligned} \tag{4.9}
$$

Taking supremum over t, we obtain $\|\mathcal{B}x_n\| = K_1 \ \forall \ n \in N$.
This shows that $\{\mathcal{B}x_n\}$ is a uniformly bounded sequence in $\mathcal{B}(B_r[0])$.

(B) Now we proceed to show that sequence $\{\mathcal{B}x_n\}$ is also equicontinuous.

Let $\varepsilon > 0$ be given. Since there is constant $T > 0$.

Case I: If $t_1, t_2 \in [0,T]$, then we have

$$
\begin{aligned}
|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1| &= \left| \frac{1}{G(\zeta)} \int_0^{t_2} \frac{g(t, \int_0^t h(s,x_s^n))}{(t_2-s)^{1-\zeta}} ds - \frac{1}{G(\zeta)} \int_0^{t_1} \frac{g(t, \int_0^t h(s,x_s^n))}{(t_1-s)^{1-\zeta}} ds \right| \\
&= \left| \frac{1}{G(\zeta)} \int_0^{t_2} \frac{|g(t, \int_0^t h(s,x_s^n))|}{(t_2-s)^{1-\zeta}} ds - \frac{1}{G(\zeta)} \int_0^{t_1} \frac{|g(t, \int_0^t h(s,x_s^n))|}{(t_1-s)^{1-\zeta}} ds \right|
\end{aligned}
$$

$$
\begin{aligned}
&= \left| \frac{1}{G(\zeta)} \int_0^{t_2} \frac{h_1(s)}{(t_2-s)^{1-\zeta}} ds - \frac{1}{G(\zeta)} \int_0^{t_1} \frac{h_1(s)}{(t_1-s)^{1-\zeta}} ds \right| \\
&= \frac{1}{G(\zeta)} \left| \int_0^{t_2} \frac{h_1(s)}{(t_2-s)^{1-\zeta}} ds - \int_0^{t_1} \frac{h_1(s)}{(t_1-s)^{1-\zeta}} ds \right| \\
&= \frac{1}{G(\zeta)} |v(t_2)-v(t_1)|
\end{aligned} \tag{4.10}
$$

from the uniform continuity of the function $v(t)$ on $[0,T]$, we get $|(\mathcal{B}x_n)t_2-(\mathcal{B}x_n)t_1| \to 0$ as $t_1 \to t_2$.

Case II: If $t_1, t_2 = T$, then we have

$$
\begin{aligned}
|(\mathcal{B}x_n)t_2-(\mathcal{B}x_n)t_1| &= \left| \frac{1}{G(\zeta)} \int_0^{t_2} \frac{g(t,\int_0^t h(s,x_s^n))}{(t_2-s)^{1-\zeta}} ds - \frac{1}{G(\zeta)} \int_0^{t_1} \frac{g(t,\int_0^t h(s,x_s^n))}{(t_1-s)^{1-\zeta}} ds \right| \\
&= \left| \frac{1}{G(\zeta)} \int_0^{t_2} \frac{|g(t,\int_0^t h(s,x_s^n))|}{(t_2-s)^{1-\zeta}} ds - \frac{1}{G(\zeta)} \int_0^{t_1} \frac{|g(t,\int_0^t h(s,x_s^n))|}{(t_1-s)^{1-\zeta}} ds \right| \\
&= \left| \frac{1}{G(\zeta)} \int_0^{t_2} \frac{|g(t,\int_0^t h(s,x_s^n))|}{(t_2-s)^{1-\zeta}} ds \right| + \left| \frac{1}{G(\zeta)} \int_0^{t_1} \frac{|g(t,\int_0^t h(s,x_s^n))|}{(t_1-s)^{1-\zeta}} ds \right| \\
&= \frac{v(t_2)}{G(\zeta)} + \frac{v(t_1)}{G(\zeta)} \\
&= 0 + \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon \text{ as } t_1 \to t_2 .
\end{aligned} \tag{4.11}
$$

Case III: If $t_1, t_2 \in \mathcal{R}_+$ with $t_1 < T < t_2$, then we have

$$
|(\mathcal{B}x_n)t_2-(\mathcal{B}x_n)t_1| = |\mathcal{B}x_n(t_2)-\mathcal{B}x_n(T)| + |\mathcal{B}x_n(T)-\mathcal{B}x_n(t_1)| . \tag{4.12}
$$

Now, if $t_1 \to t_2$ then $t_1 \to T$ and $T \to t_2$.

Therefore,

$$
|\mathcal{B}x_n(t_2)-\mathcal{B}x_n(T)| \to 0, \quad |\mathcal{B}x_n(T)-\mathcal{B}x_n(t_1)| \to 0
$$

and so

$$
|(\mathcal{B}x_n)t_2-(\mathcal{B}x_n)t_1| \to 0 \text{ as } t_1 \to t_2 \quad \text{for all } t_1, t_2 \in \mathcal{R}_+ . \tag{4.13}
$$

Hence $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in $\mathcal{B}(B_r[0])$.

Now, an application of the Arzela-Ascoli theorem yields that $\{\mathcal{B}x_n\}$ has a uniformly convergent subsequence in $\mathcal{B}(B_r[0])$ and consequently $\mathcal{B}(B_r[0])$ is a relatively compact subset of X. This shows that $\mathcal{B}$ is compact operator on $B_r[0]$. Hence by Dugungi $\mathcal{B}$ is completely continuous on $B_r[0]$.

Step IV: Next, we show that $\mathcal{A}x+\mathcal{B}$, $x \in B_r[0]$ for all $x \in B_r[0]$ is arbitrary, then

$$
x(t) = f(t,x(t)) + I^\zeta g\left(t, \int_0^t h(s,x_s)\right) ds
$$

$$
\begin{aligned}
|\mathcal{A}x(t)+\mathcal{B}x(t)| &= |\mathcal{A}x(t)| + |\mathcal{B}x(t)| \\
&= |f(t,x(t))| + \left| I^\zeta g\left(t, \int_0^t h(s,x_s)\right) ds \right|
\end{aligned}
$$

$$
\begin{aligned}
&= F + \left[\frac{1}{G(\zeta)}\int_0^t \frac{|g(t,\int_0^t h(s,x_s))|}{(t-s)^{1-\zeta}}ds\right] \\
&= F + \left[\frac{1}{G(\zeta)}\int_0^t \frac{h_1(s)}{(t-s)^{1-\zeta}}ds\right] \\
&= F + \left[\frac{v(t)}{G(\zeta)}\right] \\
&= F + [K_1] \\
&= r \quad \text{for all } t \text{ in } \mathcal{R}_+.
\end{aligned}
$$

Taking the supremum over t, we obtain $\|\mathcal{A}x+\mathcal{B}x\| = r$ for all $x \in B_r[0]$.

Hence hypothesis (C) of Theorem holds. Now, applying Krisonoselkii's Theorem 3.3 gives that QFIE (2.1) has a solution on $\mathcal{R}_+$.

Hence hypothesis (C) of Theorem 3.3 holds.

Step V: Now, for the local attractivity of the solutions for (2.1), let's assume that x and y be any two solutions of the (2.1) in $B_r[0]$ defined on $\mathcal{R}_+$. Then we have,

$$
\begin{aligned}
|x(t)-y(t)| &= \left|f(t,x(t)) + I^\zeta g\left(t,\int_0^t h(s,x_s)\right)ds - f(t,y(t)) - I^\zeta g\left(t,\int_0^t h(s,y_s)\right)ds\right| \\
&= |f(t,x(t))| + |f(t,y(t))| + \left|I^\zeta g\left(t,\int_0^t h(s,x_s)\right)ds - I^\zeta g\left(t,\int_0^t h(s,y_s)\right)ds\right| \\
&= F + F + \left|\frac{1}{G(\zeta)}\int_0^t \frac{1}{(t-s)^{1-\zeta} g(t,\int_0^t h(s,x_s))}ds\right| \\
&\quad + \left|\frac{1}{G(\zeta)}\int_0^t \frac{1}{(t-s)^{1-\zeta}} g\left(t,\int_0^t h(s,y_s)\right)ds\right| \\
&= 2F + \left[\frac{1}{G(\zeta)}\int_0^t \frac{|g(t,\int_0^t h(s,x_s))|}{(t-s)^{1-\zeta}}ds\right] + \left[\frac{1}{G(\zeta)}\int_0^t \frac{|g(t,\int_0^t h(s,y_s))|}{(t-s)^{1-\zeta}}ds\right] \\
&= 2F + \left[\frac{1}{G(\zeta)}\int_0^t \frac{h_1(s)}{(t-s)^{1-\zeta}}ds\right] + \left[\frac{1}{G(\zeta)}\int_0^t \frac{h_1(s)}{(t-s)^{1-\zeta}}ds\right] \\
&= 2F + 2\left[\frac{1}{G(\zeta)}\int_0^t \frac{h_1(s)}{(t-s)^{1-\zeta}}ds\right] \\
&= 2F + 2\left[\frac{v(t)}{G(\zeta)}\right]
\end{aligned}
\tag{4.14}
$$

for all $t \in \mathcal{R}_+$. Since and $\lim\limits_{t\to\infty} v(t) = 0$ this gives that $\limsup\limits_{t\to\infty} |x(t)-y(t)| = 0$. Thus, the (2.1) has a solution and all the solutions are locally attractive on $\mathcal{R}_+$.

5. Existence of Extremal Solution

In this section we show that given equation (2.1) has Maximal and Minimal solution:

Definition 5.1 ([33])**.** A function $f : \mathcal{R}_+ \times \mathcal{R} \times \mathcal{R}$ is called *Chandrabhan* if

(i) The function $(x,y) \to f(x,y,z)$ is measurable for each $z \in \mathcal{R}$.

(ii) The function $z \to f(x,y,z)$ is non-decreasing for almost each $(x,y) \in \mathcal{R}_+$.

Definition 5.2 ([17, 18])**.** A closed and non-empty set $\mathcal{K}$ in a Banach Algebra X is called a cone if

(i) $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$.

(ii) $\lambda\mathcal{K} \subseteq \mathcal{K}$ for $\lambda \in \mathcal{K}$, $\lambda = 0$.

(iii) $\{-\mathcal{K}\} \cap \mathcal{K} = 0$ where 0 is the zero element of X

and called positive cone if

(iv) $\mathcal{K} \circ \mathcal{K} \subseteq \mathcal{K}$

and the notation $\circ$ is a multiplication composition in X

We introduce an order relation $=$ in X as follows.

Let $x, y \in X$ then $x = y$ if and only if $y - x \in \mathcal{K}$. A cone $\mathcal{K}$ is called normal if the norm $\|\cdot\|$ is monotone increasing on $\mathcal{K}$. It is known that if the cone $\mathcal{K}$ is normal in X then every order-bounded set in X is norm-bounded set in X.

Definition 5.3. A solution x_M of the integral equation is said to be maximal if for any other solution x to the problem $x(t) = x_M(t)$ $\forall$ $t \in \mathcal{R}_+$.

Again a solution x_m of the integral equation is said to be minimal if for any other solution x to the problem $x_m(t) = x(t)$ $\forall$ $t \in \mathcal{R}_+$.

Lemma 5.1 ([9])**.** *Let $p_1, p_2, q_1, q_2 \in \mathcal{K}$ be such that $p_1 = q_1$ and $p_2 = q_2$ then $p_1p_2 = q_1q_2$. For any $p_1, p_2 \in X = \mathcal{C}(\mathcal{R}_+, \mathcal{R})$, $p_1 = p_2$ the order interval $[p_1, p_2]$ is a set in X given by, $[p_1, p_2] = \{x \in X : p_1 = x = p_2\}$.*

Definition 5.4 ([11])**.** A mapping $R : [p_1, p_2] \to X$ is said to be nondecreasing or monotone increasing if $x = y$ implies $Rx = Ry$ for all $x, y \in [p_1, p_2]$.

Theorem 5.1 ([10])**.** *Let $\mathcal{K}$ be a cone in a Banach algebra X and let $[\overline{x}, \underline{x}] \in X$. Suppose $\mathcal{A}, \mathcal{B} : [\overline{x}, \underline{x}] \to \mathcal{K}$ be two operators such that*

(a) *$\mathcal{A}$ is lipschitz with Lipschitz constant α.*

(b) *$\mathcal{B}$ is totally bounded.*

(c) $x_1 + \mathcal{B}x_2 \in [\overline{x}, \underline{x}]$ $\forall$ $x_1, x_2 \in [\overline{x}, \underline{x}]$.

(d) *$\mathcal{A}$ and $\mathcal{B}$ are nondecreasing.*

Further, if the cone $\mathcal{K}$ is positive and normal then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a least and a greatest positive solution in $[\overline{x}, \underline{x}]$ whenever $\alpha M < 1$

Where $M = \|\mathcal{B}[\overline{x}, \underline{x}]\| \sup\{\|\mathcal{B}x : x \in [\overline{x}, \underline{x}]\|\}$.

We equip the space $B\mathcal{C}(\mathcal{R}_+, \mathcal{R})$ with the order relation $=$ with the help of the cone defined by

$$\mathcal{K} = \{x \in \mathcal{C}(\mathcal{R}_+, \mathcal{R}) : x(t) = 0 \ \forall \ t \in \mathcal{R}_+\}.$$

Thus $x = \overline{x}$ iff $x(t) = \overline{x}(t)$ $\forall$ $x \in \mathcal{R}_+$.

It is well known that the cone $\mathcal{K}$ is positive and normal in $B\mathcal{C}(\mathcal{R}_+,\mathcal{R})$.

We consider another hypothesis:

(H_4): Suppose $f(t,x) = f : \mathcal{R}_+ \times \mathcal{R} \times \mathcal{R}$, $g(t,x) = g : \mathcal{R}_+ \times \mathcal{R} \times \mathcal{R}$ are Chandrabhan.

(H_5): There exists a function $\overline{h} \in L^1(\mathcal{R}_+, \mathcal{R})$ such that

$$|g(t,x)| = \overline{h}(t,x), \quad \forall\ t \in \mathcal{R}_+ \text{ and } x \in \mathcal{R}$$

(H_6): The given problem has a lower solution $\underline{x}$ and upper solution $\overline{x}$ with

$$\underline{x} = \overline{x} \ \text{ holds if } \ \|L_1\| \left\{ \frac{1}{\Gamma(\zeta+1)T^{\zeta}\|h\|_{\mathcal{L}^1}} \right\} < 1.$$

Theorem 5.2 ([10])**.** *Suppose that the Hypotheses* (H_4)-(H_6) *are holds. Then problem* (2.1) *have a minimal and maximal positive solution on* $\mathcal{R}_+$.

Proof. Let $X = B\mathcal{C}(\mathcal{R}_+,\mathcal{R})$ and define an order relation $\le$ by the cone $\mathcal{K}$ given by Definition 5.2. Clearly $\mathcal{K}$ is normal cone in X. Define the two operators $\mathcal{A}$ and $\mathcal{B}$ on X by (4.2) and (4.3) resp. then (2.1) is equivalent to operator equation $\mathcal{A}x + \mathcal{B}x = x$. Now, it is shown as in the proof, that $\mathcal{A}$ is contraction mapping and $\mathcal{B}$ is completely continuous operator.

Let $x_1, x_2 \in [\overline{x}, \underline{x}]$ s.t. $x_1 = x_2$ then by hypothesis (H_4)

$$\mathcal{A}x_1(t) = f(t,x_1(t)) = f(t,x_2(t)) = \mathcal{A}x_2(t), \quad \text{for all } t \in \mathcal{R}_+$$

and

$$\begin{aligned}
\mathcal{B}x_1(t) &= \frac{1}{G(\zeta)} \int_0^t \frac{1}{(t-s)^{1-\zeta}} g\left(t, \int_0^t h(s,x_{1_s})\right) ds \\
&= \left[\frac{1}{G(\zeta)} \int_0^t \frac{1}{(t-s)^{1-\zeta}} g\left(t, \int_0^t h(s,x_{2_s})\right) ds \right] \\
&= \mathcal{B}x_2(t).
\end{aligned}$$

So $\mathcal{A}$ and $\mathcal{B}$ are non decreasing operator on $[\overline{x}, \underline{x}]$.

Again by hypothesis

$$\begin{aligned}
\underline{x}(t) = f(t,\underline{x}(t)) &+ \left[\frac{1}{G(\zeta)} \int_0^t \frac{g(t, \int_0^t h(s,\underline{x}_s))}{(t-s)^{1-\zeta}} ds \right] \\
&= f(t,x(t)) + \left[\frac{1}{G(\zeta)} \int_0^t \frac{g(t, \int_0^t h(s,x_s))}{(t-s)^{1-\zeta}} ds \right] \\
&= f(t,\overline{x}(t)) + \left[\frac{1}{G(\zeta)} \int_0^t \frac{g(t, \int_0^t h(s,\overline{x_s}))}{(t-s)^{1-\zeta}} ds \right] \\
&= \overline{x}(t), \quad \text{for all } x \in [\overline{x}, \underline{x}] \text{ and } t \in \mathcal{R}_+.
\end{aligned}$$

Hence $\mathcal{A}x + \mathcal{B}x \in [\overline{x}, \underline{x}]$ for all $x \in [\overline{x}, \underline{x}]$

$$\begin{aligned}
M &= \|\mathcal{B}([\overline{x}, \underline{x}])\| \\
&= \sup\{\|\mathcal{B}x\| : x \in [\overline{x}, \underline{x}]\}
\end{aligned}$$

$$= \sup\left\{\frac{1}{G(\zeta)}\int_0^t \frac{g(t,\int_0^t h(s,x_s))}{(t-s)^{1-\zeta}}ds\right\}$$

$$= \frac{1}{G(\zeta)}\left|\int_0^t \frac{|g(t,\int_0^t h(s,x_s))|}{(t-s)^{1-\zeta}}ds\right|$$

$$= \frac{\|h_1\|}{G(\zeta)}\int_0^t \frac{1}{(t-s)^{1-\zeta}}ds$$

$$= \frac{\|h_1\|}{\Gamma(\zeta)}\left[\frac{(t-s)^{\zeta}}{-\zeta}\right]_0^t$$

$$= \frac{1}{\Gamma(\zeta+1)}T^{\zeta}\|h_1\|_{\mathcal{L}^1}$$

$$L_1M = \|L_1\|\left\{\frac{1}{\Gamma(\zeta+1)}T^{\zeta}\|h_1\|_{\mathcal{L}^1}\right\} < 1$$

Thus operator equation has minimal and maximal solution in $[\overline{x},\underline{x}]$.

Thus given problem have minimal and maximal positive solution on $\mathcal{R}_+$. □

Consider the following quadratic functional integral equation of type (2.1)

$$x(t) = \frac{1}{6}\{\sin 2t[x(t)]e^{-t}\} + \left[\frac{1}{G(\zeta)}\int_0^t \frac{1}{(t-s)^{1-\zeta}}\frac{1}{2t^{\zeta+5}}ds\right], \quad \forall\ t \in \mathcal{R}_+ \tag{5.1}$$

$$\int_0^t h(s,x_s)ds = \int_0^t \frac{1}{t^3}s\,ds = \frac{1}{t^3}\left[\frac{s^2}{2}\right]_0^t = \frac{1}{t^3}\frac{t^2}{2} = \frac{1}{2t}$$

$$g\left(t,\int_0^t h(s,x_s)ds\right) = \frac{1}{t^{\zeta+4}}\frac{1}{2t} = \frac{1}{2t^{\zeta+5}}$$

$(\mathcal{H}_1)$: Now

$$\begin{aligned}|f(t,x(t))-f(t,y(t))| &= \frac{1}{6}|\{\sin 2t[x(t)]e^{-t}\}-\{\sin 2t[y(t)]e^{-t}\}|\\ &= \frac{1}{6}e^{-t}|\sin 2t[x(t)-y(t)]|\\ &= l(t)|x(t)-y(t)|l(t)\\ &= \frac{1}{6}e^{-t}\sin 2t\end{aligned}$$

$(\mathcal{H}_2)$: Take $h_1(t) = \frac{1}{t^{\zeta+5}}s^2$ it is continuous on $\mathcal{R}_+$.

Implies $g\left(t,\int_0^t h(s,x_s)ds\right) = h_1(t)$.

That is $\dfrac{1}{2t^{\zeta+5}} = \dfrac{1}{t^{\zeta+5}}s^2$.

$(\mathcal{H}_3)$:

$$\begin{aligned}v(t) &= \int_0^t \frac{h_1(s)}{(t-s)^{1-\zeta}}ds\\ &= \int_0^t \frac{\frac{1}{t^{\zeta+5}}s^2}{(t-s)^{1-\zeta}}ds\\ &= \frac{1}{t^{\zeta+5}}\int_0^t \frac{s^2}{(t-s)^{1-\zeta}}ds\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{t^{\zeta+5}} \int_0^t s^2 (t-s)^{\zeta-1} ds \\
&= \frac{1}{t^{\zeta+5}} \left\{ \left[s^2 \frac{(t-s)^\zeta}{-\zeta} \right]_0^t - \int_0^t 2s \frac{(t-s)^\zeta}{-\zeta} ds \right\} \\
&= \frac{1}{t^{\zeta+5}} \left\{ 0 - \int_0^t 2s \frac{(t-s)^\zeta}{-\zeta} ds \right\} \\
&= 2 \frac{1}{t^{\zeta+5}} \int_0^t s \frac{(t-s)^\zeta}{\zeta} ds \\
&= 2 \frac{1}{t^{\zeta+5}} \left\{ \left[s \frac{(t-s)^{\zeta+1}}{-\zeta(\zeta+1)} \right]_0^t - \int_0^t \frac{(t-s)^{\zeta+1}}{-\zeta(\zeta+1)} \right\} \\
&= 2 \frac{1}{t^{\zeta+5}} \left\{ 0 - \int_0^t \frac{(t-s)^{\zeta+1}}{-\zeta(\zeta+1)} \right\} \\
&= 2 \frac{1}{t^{\zeta+5}} \left\{ \int_0^t \frac{(t-s)^{\zeta+1}}{\zeta(\zeta+1)} \right\} \\
&= 2 \frac{1}{t^{\zeta+5}} \left[\frac{(t-s)^{\zeta+2}}{-\zeta(\zeta+1)(\zeta+2)} \right]_0^t \\
&= 2 \frac{1}{t^{\zeta+5}} \left\{ 0 - \frac{t^{\zeta+2}}{-\zeta(\zeta+1)(\zeta+2)} \right\} \\
&= 2 \frac{1}{t^{\zeta+5}} \left\{ \frac{t^{\zeta+2}}{\zeta(\zeta+1)(\zeta+2)} \right\} \to 0 \text{ as } t \to \infty.
\end{aligned}
$$

$v(t)$ is continuous and bounded on $\mathcal{R}_+$ and vanish at infinity.

It follows that all the conditions $(\mathcal{H}_1)$-$(\mathcal{H}_5)$ are satisfied. Thus by Theorem 4.1, above problem (5.1) has a solution $\mathcal{R}_+$.

6. Conclusion

In this paper we have proved existence of solution to a Quadratic Functional Integro-Differential Equation of Fractional order and finally we obtained the result for extremal solution with concrete example.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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