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EXISTENCE THE SOLUTION OF COUPLED SYSTEM OF QUADRATIC HYBRID FUNCTIONAL INTEGRAL EQUATION IN BANACH ALGEBRAS

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Abstract

In this paper we prove the existence of solution of coupled system of quadratic hybrid functional integral equations. Our main result is based on the standard tools of fixed point theory. The Existence and locally attractivity is proved in $\mathcal{R}_+$.

Keywords: Quadratic Hybrid Functional Integral Equations, Banach Algebras, R-L Fractional Derivative, Hybrid FPT, Existence result.

I. Introduction

The fractional calculus is the more effective and powerful tool for the Mathematical modeling of several engineering and scientific Phenomenon. For instance they appear in the fluid dynamics or study of air modulation, electromagnetism, electricity, or the control of nonlinear process. The combination of fractional calculus and integral equations may introduce more effective tool for analysis.

In the nonlinear analysis the dynamical systems represented by nonlinear functional differential and integral equations are simplified by Perturbation Techniques. Hybrid differential equations and hybrid integral equations are important type of the perturbed differential and integral equations.

Hybrid fractional integral equations have also been studied by several researchers. [VIII, XII, XIX-XXV] contains results related to hybrid fractional differential and integral equations. In order to achieve various goals, some researchers have obtained the lot of useful and interesting coupled system of functional differential and integral equations during the past few years. Major times hybrid fixed point theory can be used for developing the existence theory for hybrid equations. Recently, Su [XXXV] analyzed a two point boundary value problem corresponding to a coupled system of fractional differential equations. Also Gafiychuk et al. [XIV] discussed the solutions of particular case, namely the coupled nonlinear fractional reaction-diffusion equations.

Here we study the existence results for coupled system of quadratic hybrid functional integral equations in Banach Algebras along with the locally attractivity.

II. Statement of the problem:

Let $\zeta \in (0,1)$ $\mathcal{R}$denote the real numbers whereas $\mathcal{R}_+$ be the set of nonnegative numbersi.e. $\mathcal{R}_+ = [0,\infty) \subset \mathcal{R}$

Consider the Coupled system of quadratic hybrid functional integral equations

$$\left.\begin{aligned} x(t) &= \left[q(t) + f(t, y(\mu(t)))\right] \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, y(\gamma(t))\right)}{(t-s)^{1-\zeta}} ds \ \forall t \in \mathcal{R}_+ \\ y(t) &= \left[q(t) + f(t, x(\mu(t)))\right] \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, x(\gamma(t))\right)}{(t-s)^{1-\zeta}} ds \ \forall t \in \mathcal{R}_+ \end{aligned}\right\} \tag{2.1}$$

Where $\zeta \in (0,1)$and the functions$f(t,x) = f: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}, g: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}\mu, \gamma: \mathcal{R}_+ \to \mathcal{R}_+$, are continuous.

By a solution of system of CQHFIE (2.1) we mean a function $(x,y) \in BC(\mathcal{R}_+, \mathcal{R} \times \mathcal{R})$is the space of continuous and bounded real-valued functions defined on$\mathcal{R}_+$ such that

i) The function

$$t \to \frac{x(t)}{q(t) + f(t, y(\mu(t)))}$$

is continuous and bounded for each $x \in \mathcal{R}$

ii) (x,y)Satisfies the system of equations in (2.1)

Firstly we convert the CQHFIE (2.1) into an operator equation and then apply the coupled fixed point theorem [XXXVII] in the Banach space.

This section is devoted to collecting some definitions and results which will be needed further.

III. Preliminaries:

Let $X = BC(\mathcal{R}_+, \mathcal{R})$be Banach algebra with norm $\|.\|$ and letΩ be a subset of X. Let a mapping$\mathcal{A} : X \to X$ be an operator and consider the following operator equation in X, namely,

$$x(t) = (\mathcal{A}x)(t) t \in \mathcal{R}_+ \tag{3.1}$$

Definition 3.1[XXXVI, XXXVII]: An element $(x,y) \in X \times X$ is called coupled fixed point of a mapping $\mathrm{T}: X \times X \to X$ if$\mathrm{T}(x,y) = x$and $\mathrm{T}(y,x) = y$

Definition 3.2[XXIX]: The solution $x(t)$ of the equation (2.1) is said to be locally attractive if there exists an closed ball $B_r[0]$ in$BC(\mathcal{R}_+, \mathcal{R})$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (2.1) belonging to $B_r[0] \cap \Omega$suchthat

$$\lim_{t\to\infty}\left(x(t) - y(t)\right) = 0 \tag{3.2}$$

Definition 3.3[XXIX]: Let X be a Banach space. A mapping $\mathcal{A} : X \to X$is called Lipschitz if there is a constant$\alpha > 0$ such that$\|\mathcal{A}x - \mathcal{A}y\| \leq \alpha\|x - y\|$ for all$x, y \in X$ If$\alpha < 1$ then $\mathcal{A}$is called a contraction on X with the contraction constant α.

Definition 3.4: (Dugundji and Granas [XXIII]). An operator $\mathcal{A}$ on a Banach space X into itself is called Compact if for any bounded subset S of X, $\mathcal{A}$(S) is a relatively compact subset of X. If $\mathcal{A}$ is continuous and compact, then it is called completely continuous on X.

Let X be a Banach space with the norm $\|.\|$and Let$\mathcal{A} : X \to X$be an operator (in general nonlinear). Then $\mathcal{A}$ is called

1) Compact if $\mathcal{A}(X)$ is relatively compact subset of X;
2) Totally Compact if $\mathcal{A}(S)$is a totally bounded subset of X for any bounded subset S of X
3) Completely continuous if it is continuous and totally bounded operator on X.

It is clear that every compact operator is totally bounded but the converse need not be true.

The solutions of (2.1) in the space $BC(\mathcal{R}_+, \mathcal{R})$of continuous and bounded real-valued functions defined on$\mathcal{R}_+$. Define a standard supremum norm $\|.\|$ and a multiplication "." in $BC(\mathcal{R}_+, \mathcal{R})by\|x\| = \sup\{|x(t)|: t \in \mathcal{R}_+\}$ (3.3)

$$(xy)(t) = x(t)y(t) \quad t \in \mathcal{R}_+ \tag{3.4}$$

Clearly, $BC(\mathcal{R}_+, \mathcal{R})$becomes a Banach space with respect to the above norm and the multiplication in it. By $\mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ we denote the space of Lebesgue-integrable functions on $\mathcal{R}_+$ with the norm$\|.\|_{\mathcal{L}^1}$ defined by

$$\| x\|_{\mathcal{L}^1} = \int_0^\infty |x(t)|dt \tag{3.5}$$

Denote by $\mathcal{L}^1(a, b)$be the space of Lebesgue-integrable functions on the interval (a,b), which is equipped with the standard norm. Let $x \in \mathcal{L}^1(a, b)$ and let $\beta > 0$ be a fixed number.

Definition 3.5[XVIII]: The Riemann-Liouville fractional integral of order $\beta$of the function $x(t)$ is defined by the formula:

$$I^\beta x(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{x(s)}{(t-s)^{1-\beta}} ds \quad t \in (a, b) \tag{3.6}$$

Where $\Gamma(\beta)$ denote the gamma function.

It may be shown that the fractional integral operator I^β transforms the space $\mathcal{L}^1(a, b)$ into itself and has some other properties (see [12-19])

Definition 3.6: A set A⊆ [a,b] is said to be measurable if $m^* A = m_* A$. In this case we define *mA*, the measure of A as *mA* = $m^* A = m_* A$

If $A_1$and A_2 are measurable subsets of [a,b] then their union and their intersection is also measurable.

Clearly every open or closed set in R is measurable.

Definition 3.7: Let f be a function defined on [a,b].Then f is measurable function if for each $\alpha \in R$, $the set$ $\{x: f(x) > \alpha\}$ is measurable set .

i.e. f is measurable function if for every real number $\alpha$the inverse image of (α, ∞) is an open set

As (α, ∞) is an open set and if f is continuous, then inverse image under of $(\alpha, \infty)$is open. Open sets being measurable, hence every continuous function is measurable.

Definition 3.8: A sequence of functions $\{f_n\}$is said to converge uniformly on an interval $[a, b]$ to a function f if for any $\epsilon > 0$ and for all $x \in [a, b]$ there exists an integer N (dependent only on ϵ) such that for all $x \in [a, b]$

$$|f_n(x) - f(x)| < \epsilon \quad \forall\, n \geq N$$

Definition 3.9: The Family F is equicontinuous at a point $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(f(x_0), f(x)) < \varepsilon$ for all $f \in F$and all x such that $d(x_0, x) < \delta$.

The family is point-wise equicontinuous if it is equicontinuous at each point of X.

The family is uniformly equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ a such that $d(f(x_1), f(x_2)) < \varepsilon$ for all $f \in F$and all $x_1, x_2 \in X$ such that $d(x_1, x_2) < \delta$.

Definition 3.10: Let$X = \mathcal{C}(\mathcal{R}_+, \mathcal{R})$equipped with the supremum norm. Clearly it is a Banach space with respect to pointwise operations and the supremum norm.

Define scalar multiplication and a sum on $X \times X$ as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and $\theta(x, y) = (\theta x, \theta y)for\theta \in \mathcal{R}$. Then$X \times X$ is a vector space on$\mathcal{R}$.

Lemma 3.1[XXXVII]: Let$S$ be a non -empty, convex, closed and bounded subset of the Banach space X and let $\tilde{S} = S \times S$suppose that $\mathcal{A}, \mathcal{C}: X \to X$ and $B: S \to X$ are three operators satisfying:

a) $\mathcal{A}$,$\mathcal{C}$ are Contraction with a Contraction constant $\alpha, \beta < 1$,
b) $\mathcal{B}$ is completely continuous, and
c) $x = \mathcal{A}x.\mathcal{B}y + \mathcal{C}x \in S$ for all $x, y \in S$
d) $\xi M + \eta < 1$ where $M = \|\mathcal{B}(s)\| = sup\{\|\mathcal{B}x\|: x \in S\}$

Then the operator equation $T(x, y) == \mathcal{A}x.\mathcal{B}y + \mathcal{C}x$has at least a coupled fixed point solution in$\tilde{S}$.

Lemma 3.2[XXXVI, XXXVII]: Let$\tilde{X} = X \times X$. Define $\|(x, \mathrm{y})\| = \|x\| + \|\mathrm{y}\|$

Then $\tilde{X}$ is Banach space with respect to the above norm.

Lemma 3.3 [**XXI**]: Let$q > 0 and x \in \mathcal{C}(0, T) \cap \mathcal{L}(0, T)$ then we have

$$I^q \frac{d^q}{dt^q} x(t) = x(t) - \sum_{j=1}^{n} \frac{(I^{n-q}x)^{(n-j)}(0)}{\Gamma(q-j+1)} t^{q-j}$$

Where$n - 1 < q < n$.

Now we introduce a coupled fixed point theorem which is generalization of lemma (3.1) .We use the following fixed point theorem for proving existence the solution of the system of CQHFIE (2.1) under certain Caratheodory condition.

Theorem 3.1 [XXXVI,XXXVII]: Let$S$ be a non-empty, convex, closed and bounded subset of the Banach space X and $\tilde{S} = S \times S$, let $\mathcal{A}: X \to X$ and $B: S \to X$ are two operators satisfying:

a) $\mathcal{A}$ is Lipschitzian with a Lipschitz constant α,
b) $\mathcal{B}$ is completely continuous, and

c) $x = \mathcal{A}x.\mathcal{B}y \in S, x \in S$ for all $y \in S$

d) $\xi M < 1$ where $M = \|\mathcal{B}(s)\| = sup\{\|\mathcal{B}x\|: x \in S\}$

Then the operator equation $T(x,y) = \mathcal{A}x.\mathcal{B}y$ has at least a coupled fixed point solution in $\tilde{S}$.

Theorem 3.2: (Arzela-Ascoli theorem (XII)): Every uniformly bounded and equi-continuous sequence $\{ f_n \}$ of functions in $\mathcal{C}(J,\mathcal{R})$, has a convergent subsequence.

Theorem 3.3[XII]: A metric space X is compact iff every sequence in X has a convergent subsequence.

IV. Existence results

Definition 4.1[XII]: A mapping $g: \mathcal{R}_+ \times \mathcal{R} \rightarrow \mathcal{R}$ is said to be Caratheodory if

1. $t \rightarrow g(t,x)$ is measurable for all $x \in \mathcal{R}$, and
2. $x \rightarrow g(t,x)$ is continuous almost everywhere for $t \in \mathcal{R}_+$

   Again a caratheodory function g is called $\mathcal{L}^1$-Caratheodory if

3. for each real number $r > 0$ there exists a function $h_r \in \mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ such that $|g(t,x)| \leq h_r(t)$ a.e. $t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$ with $|x| \leq r$

Finally, a Caratheodory function $g(t,x)$ is called $\mathcal{L}^1_{\mathcal{R}}$ − Caratheodory if

4. there exist a function $h \in \mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ such that $|g(t,x)| \leq h(t)$

   a.e. $t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$

For convenience, the function h is referred to as a bound function of g.

We consider the coupled nonlinear hybrid Functional integral equation (2.1) assuming that the following hypothesis are satisfied.

(H_0) $\mu, \gamma: \mathcal{R}_+ \rightarrow \mathcal{R}_+$, are continuous and $q(t): \mathcal{R}_+ \rightarrow \mathcal{R}$ such that $Q = \underbrace{sup}_{t \in \mathcal{R}_+} |q(t)|$

(H_1)The function $f_i(t,x): \mathcal{R}_+ \times \mathcal{R} \rightarrow \mathcal{R}$ is continuous and bounded with bound

$F = \underbrace{sup}_{(t,x) \in \mathcal{R}_+ \times \mathcal{R}} |f_i(t,x)|$ there exists a bounded function $l: \mathcal{R}_+ \rightarrow \mathcal{R}_+$ with bound L Satisfying $|f_i(t,x) - f_i(t,y)| \leq L\,|\,x(t) - y(t)| t \in \mathcal{R}_+$ for all $x, y \in \mathcal{R}$

(H_2)The functions $g_i: \mathcal{R}_+ \times \mathcal{R} \rightarrow \mathcal{R}$ satisfy caratheodory condition (i.e.measrable in t for all $x \in \mathcal{R}$ and continuous in x for all $t \in \mathcal{R}_+$) and there exist function $m_1 \in \mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ Such that $g_i(t,x) \leq m_1(t) \forall (t,x) \in \mathcal{R}_+ \times \mathcal{R}$

(H_3)The uniform continuous function $v: \mathcal{R}_+ \rightarrow \mathcal{R}_+$ defined by the formulas

$$v(t) = \int_0^t \frac{m_1(s)}{(t-s)^{1-\zeta}} \, ds ,$$

is bounded on $\mathcal{R}_+$ and vanishat infinity, that is, $\lim_{t \to \infty} v(t) = 0$

Remark 4.1: Note that if the hypothesis (H_2) hold, then there exist constants $K_1 > 0$ and such that:

$$K_1 = \underbrace{sup}_{t\geq 0}\frac{1}{\Gamma(\zeta)}\int_0^t \frac{m_1\,(s)}{(t-s)^{1-\zeta}}\;ds$$

Theorem 4.1: Suppose that the hypotheses $[\,(H_1)$-$(H_5)]$ are hold. Furthermore if$(\|\alpha\|K_1) < 1$ where K_1 are defined remark (4.1), Then the equation (2.1) has a solution in the space $BC(\mathcal{R}_+\,,\mathcal{R})$.Moreover, solutions of the equation (2.1) are locally attractive on$\mathcal{R}_+$.

Proof: Let $X = BC(\mathcal{R}_+\,,\mathcal{R})$be Banach Algebras of all continuous and bounded real valued function on $\mathcal{R}_+$ with the norm$\|x\| = \underbrace{sup}_{t\in\,\mathcal{R}_+}\,|x(t)|$

We show that existence of solution for (2.1) under some suitable conditions on the functions involved in (2.1).

Set$X = C(\mathcal{R}_+,\mathcal{R})$ and a Closed subset $B_r[0]$ of Xcentered at origin 0 and of radius r, defined by

$$B_r[0] = \{x \in X\;|\|x\| \leq r\},$$

$$where\;[\|Q\| + \|F\|]\|m_1\|\frac{T^\zeta}{\Gamma(\zeta+1)} = r > 0$$

Clearly $S = B_r[0]$ be a non -empty, convex, closed and bounded subset of the Banach space X.

Define two operators $\mathcal{A}: X \to X$and$\mathcal{B}: B_r[0] \to X$ by

$$\mathcal{A}x(t) = [q(t) + f(t, x(\mu(t)))] \tag{4.2}$$

$$\mathcal{B}x(t) = \frac{1}{\Gamma(\zeta)}\int_0^t \frac{g\Big(t, x\big(\gamma(t)\big)\Big)}{(t-s)^{1-\zeta}}\;ds \tag{4.3}$$

So the system (2.1) transformed into the system of operator equations as

$$\left.\begin{aligned} x(t) &= \mathcal{A}y(t)\mathcal{B}y(t)\\ y(t) &= \mathcal{A}x(t)\mathcal{B}x(t)\end{aligned}\right\}\qquad t \in \mathcal{R}_+,$$

It is sufficient to prove $\mathcal{A}(x,y\,)\,\mathcal{B}(x,y) = (x,y)$ has at least one solution, because

$$\begin{aligned}&\big(T(x,y), T(y,x)\big) = (\mathcal{A}x\mathcal{B}y, \mathcal{A}y\mathcal{B}x)\\ &= (\mathcal{A}x, \mathcal{A}y\,)\,(\mathcal{B}y, \mathcal{B}x)\\ &= \mathcal{A}(x,y\,)\,\mathcal{B}(y,x)\\ &= (x,y)\end{aligned} \tag{4.4}$$

Which implies that $T(x,y)$has at least one coupled fixed point.

Therefore$\mathcal{A}$,$\mathcal{B}$define the operators $\mathcal{A},\mathcal{B}: B_r[0] \to X$.

we wish to show that $\mathcal{A}$,$\mathcal{B}$satisfy all the requirements of theorem (3.1) on $B_r[0]$.

Step I: Firstly, we show that $\mathcal{A}$ is Lipschitz on .Let$x, y \in X$ be arbitrary, and then by hypothesis(H_1), we get

$$\begin{aligned}|\mathcal{A}x(t) - \mathcal{A}y(t)| &= \big|q(t) + f(t, x(\mu(t))) - \big(q(t) + f(t, y(\mu(t)))\big)\big|\\ &= \big|f(t, x(\mu(t)) - f(t, y(\mu(t))\big|\\ &\leq \boldsymbol{l}\|x-y\|\text{for all } t \in \mathcal{R}_+\end{aligned}$$

Taking supremum over t

$\|\mathcal{A}x - \mathcal{A}y\| \leq \boldsymbol{L}\|x - y\|$for all $x, y \in X$ (4.5)

This shows that $\mathcal{A}$ is Lipschitzian on$X$ with the Lipschitz constant $\boldsymbol{L}$.

Step II: Secondly, To Prove the operator$\mathcal{B}$ is completely continuous operator ($\mathcal{B}$ is compact and continuous operator) on$B_r[0]$.

Case I) Firstly we show that $\mathcal{B}$ is continuous on $B_r[0]$**.**

Let by dominated convergence theorem, let $\{x_n\}$ be a sequence in S such that $\{x_n\} \to x$.Then,

$$\lim_{n\to\infty} \mathcal{B}x_n(t) = \lim_{n\to\infty}\left\{\frac{1}{\Gamma(\zeta)}\int_0^t \frac{g\left(t, x_n\big(\gamma(t)\big)\right)}{(t-s)^{1-\zeta}} ds \right\}$$

$$= \frac{1}{\Gamma(\zeta)}\int_0^t \frac{g\left(t, x\big(\gamma(t)\big)\right)}{(t-s)^{1-\zeta}} ds$$

$$= \mathcal{B}x(t), \qquad \forall t \in \mathcal{R}_+$$

This shows that $\mathcal{B}x_n$ converges to$\mathcal{B}x$ pointwise in$S$.

Next to show that sequence $\{\mathcal{B}x_n\}$ is equicontinuous sequence in$S$.

Let $t_1, t_2 \in \mathcal{R}_+$be arbitrary with $t_1 < t_2$ then

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| = \left| \begin{array}{c} \frac{1}{\Gamma(\zeta)}\int_0^{t_2} \frac{g\left(t_2, x_n\big(\gamma(s)\big)\right)}{(t_2-s)^{1-\zeta}} ds - \\ \frac{1}{\Gamma(\zeta)}\int_0^{t_1} \frac{g\left(t_1, x_n\big(\gamma(s)\big)\right)}{(t_1-s)^{1-\zeta}} ds \end{array} \right|$$

$$\leq \frac{1}{\Gamma(\zeta)}\left|\int_0^{t_2}(t_2-s)^{\zeta-1} g\left(t_2, x_n\big(\gamma(s)\big)\right) ds - \int_0^{t_2}(t_1-s)^{\zeta-1} g\left(t_1, x_n\big(\gamma(s)\big)\right) ds\right|$$

$$+\frac{1}{\Gamma(\zeta)}\left|\int_0^{t_2}(t_1-s)^{\zeta-1} g\left(t_1, x_n\big(\gamma(s)\big)\right) ds - \int_0^{t_1}(t_1-s)^{\zeta-1} g\left(t_1, x_n\big(\gamma(s)\big)\right) ds\right|$$

$$\leq \frac{1}{\Gamma(\zeta)}\left|\int_0^{t_2}(t_2-s)^{\zeta-1} m_1(s) ds - \int_0^{t_2}(t_1-s)^{\zeta-1} m_1(s) ds\right|$$

$$+\frac{1}{\Gamma(\zeta)}\left|\int_0^{t_2}(t_1-s)^{\zeta-1} m_1(s) ds - \int_0^{t_1}(t_1-s)^{\zeta-1} m_1(s) ds\right|$$

$$\leq \frac{\|m_1\|_{\mathcal{L}^1}}{\Gamma(\zeta)}\left\{\left|\int_0^{t_2}\left[(t_2-s)^{\zeta-1} - (t_1-s)^{\zeta-1}\right] ds\right| + \left|\int_{t_1}^{t_2}(t_1-s)^{\zeta-1} ds\right|\right\}$$

$$\leq \frac{\|m_1\|_{\mathcal{L}^1}}{\Gamma(\zeta)}\left\{\left|\left[\frac{(t_2-s)^\zeta}{-\zeta}\right]_0^{t_2}-\left[\frac{(t_1-s)^\zeta}{-\zeta}\right]_0^{t_2}\right|+\left|\left[\frac{(t_1-s)^\zeta}{-\zeta}\right]_{t_1}^{t_2}\right|\right\}$$

$$\leq \frac{\|m_1\|_{\mathcal{L}^1}}{\zeta\Gamma(\zeta)}\left\{\begin{matrix}\left|\left[(t_2-t_2)^\zeta-(t_2-0)^\zeta\right]-\left[(t_1-t_2)^\zeta-(t_1-0)^\zeta\right]\right| \\ +\left|(t_1-t_2)^\zeta-(t_1-t_1)^\zeta\right|\end{matrix}\right\}$$

$$\leq \frac{\|m_1\|_{\mathcal{L}^1}}{\Gamma(\zeta+1)}\left\{\left|(t_1)^\zeta-(t_2)^\zeta-(t_1-t_2)^\zeta\right|+\left|(t_1-t_2)^\zeta\right|\right\}$$

$\to 0$ as$t_1 \to t_2, \forall n \in \mathcal{N}$

This shows that the convergence is uniform, by using property of uniform convergence that is uniform convergence imply continuity.

Hence $\mathcal{B}$ is continuous on $B_r[0]$.

Case II): To show $\mathcal{B}$ is compact operator on $B_r[0]$, for this to show that $\mathcal{B}$ is uniformly bounded and equicontinuous in $B_r[0]$.
First we show that $\mathcal{B}$ is uniformly bounded. Let $x \in S$ be arbitrary then

$$|\mathcal{B}x(t)| = \left|\frac{1}{\Gamma(\zeta)}\int_0^t \frac{g\left(t,x(\gamma(s))\right)}{(t-s)^{1-\zeta}}ds\right|$$

$$\leq \frac{1}{\Gamma(\zeta)}\int_0^t \frac{\left|g\left(t,x(\gamma(s))\right)\right|}{(t-s)^{1-\zeta}}ds$$

$$\leq \frac{1}{\Gamma(\zeta)}\int_0^t \frac{m_1(s)}{(t-s)^{1-\zeta}}ds$$

$$\leq \frac{1}{\Gamma(\zeta)}v(t)$$

Taking supremum over t, we obtain

$$\|\mathcal{B}x\| \leq \frac{v(t)}{\Gamma(\zeta)} = K_1, \forall t \in \mathcal{R}_+$$

Hence $\mathcal{B}$is uniformly bounded subset of $B_r[0]$.
Now to show $\mathcal{B}$ is equicontinuous on $B_r[0]$.
Let $t_1, t_2 \in \mathcal{R}_+$, then

$$|\mathcal{B}x(t_2)-\mathcal{B}x(t_1)| = \left|\begin{matrix}\frac{1}{\Gamma(\zeta)}\int_0^{t_2}(t_2-s)^{\zeta-1}g\left(t_2,x(\gamma(s))\right)ds - \\ \frac{1}{\Gamma(\zeta)}\int_0^{t_1}(t_1-s)^{\zeta-1}g\left(t_1,x(\gamma(s))\right)ds\end{matrix}\right|$$

$$\leq \frac{1}{\Gamma(\zeta)}\left|\int_0^{t_2}(t_2-s)^{\zeta-1}g\left(t_2,x\big(\gamma(s)\big)\right)ds-\int_0^{t_2}(t_1-s)^{\zeta-1}g\left(t_1,x\big(\gamma(s)\big)\right)ds\right|$$

$$+\frac{1}{\Gamma(\zeta)}\left|\int_0^{t_2}(t_1-s)^{\zeta-1}g\left(t_1,x\big(\gamma(s)\big)\right)ds-\int_0^{t_1}(t_1-s)^{\zeta-1}g\left(t_1,x\big(\gamma(s)\big)\right)ds\right|$$

$$\leq \frac{1}{\Gamma(\zeta)}\left|\int_0^{t_2}(t_2-s)^{\zeta-1}m_1(s)ds-\int_0^{t_2}(t_1-s)^{\zeta-1}m_1(s)ds\right|$$

$$+\frac{1}{\Gamma(\zeta)}\left|\int_0^{t_2}(t_1-s)^{\zeta-1}m_1(s)ds-\int_0^{t_1}(t_1-s)^{\zeta-1}m_1(s)ds\right|$$

$$\leq \frac{\|m_1\|_{\mathcal{L}^1}}{\Gamma(\zeta)}\left\{\left|\int_0^{t_2}\left[(t_2-s)^{\zeta-1}-(t_1-s)^{\zeta-1}\right]ds\right|+\left|\int_{t_1}^{t_2}(t_1-s)^{\zeta-1}ds\right|\right\}$$

$$\leq \frac{\|m_1\|_{\mathcal{L}^1}}{\Gamma(\zeta)}\left\{\left|\left[\frac{(t_2-s)^{\zeta}}{-\zeta}\right]_0^{t_2}-\left[\frac{(t_1-s)^{\zeta}}{-\zeta}\right]_0^{t_2}\right|+\left|\left[\frac{(t_1-s)^{\zeta}}{-\zeta}\right]_{t_1}^{t_2}\right|\right\}$$

$$\leq \frac{\|m_1\|_{\mathcal{L}^1}}{\zeta\Gamma(\zeta)}\left\{\begin{matrix}\left|\left[(t_2-t_2)^{\zeta}-(t_2-0)^{\zeta}\right]-\left[(t_1-t_2)^{\zeta}-(t_1-0)^{\zeta}\right]\right| \\ +\left|(t_1-t_2)^{\zeta}-(t_1-t_1)^{\zeta}\right|\end{matrix}\right\}$$

$$\leq \frac{\|m_1\|_{\mathcal{L}^1}}{\Gamma(\zeta+1)}\left\{\left|(t_1)^{\zeta}-(t_2)^{\zeta}-(t_1-t_2)^{\zeta}\right|+\left|(t_1-t_2)^{\zeta}\right|\right\}$$

$$\to 0\ as t_1 \to t_2.$$

Implies that $\mathcal{B}$ is equicontinuous.Hence $\mathcal{B}$ is compact subset of $B_r[0]$.
Implies that $\mathcal{B}$ is completely continuous on $B_r[0]$.
Therefore, it follows from Arzela-Ascoli theorem $\mathcal{B}$ is completely continuous on $B_r[0]$.
Step III: Next we show that $x(t) = \mathcal{A}x.\mathcal{B}y \in B_r[0]$ for all $x, y \in B_r[0]$ is arbitrary, then

$$|x(t)| = |\mathcal{A}y(t).\mathcal{B}y(t)| \leq |\mathcal{A}y(t)|.|\mathcal{B}y(t)|$$

$$\leq \left|q(t)+f\big(t,y(\mu(t))\big)\right|\left|\frac{1}{\Gamma(\zeta)}\int_0^t\frac{g\left(t,y\big(\gamma(s)\big)\right)}{(t-s)^{1-\zeta}}ds\right|$$

$$\leq \left[|Q|+\left|f\big(t,y(\alpha(t))\big)\right|\right]\left[\frac{1}{\Gamma(\zeta)}\int_0^t\frac{\left|g\left(t,y\big(\gamma(t)\big)\right)\right|}{(t-s)^{1-\zeta}}ds\right]$$

$$\leq [|Q|+|F|]\left[\frac{1}{\Gamma(\zeta)}\int_0^t\frac{1}{(t-s)^{1-\zeta}}|m_1(t)|\,ds\right]$$

$$\leq [\|Q\|+\|F\|]\|m_1\|\frac{T^{\zeta}}{\Gamma(\zeta+1)}=r$$

Taking the supremum over t, we obtain $\| \mathcal{A}y\mathcal{B}y \| \leq r$ for all $x, y \in \; B_r[0]$
Hence hypothesis (c) of Theorem (3.1) holds.

Also we have $M = \| \mathcal{B}(B_r[0]) \; \| = \sup\{ \| \mathcal{B}x \| : \; x \in B_r[0]$

$$= sup\{ \underbrace{sup}_{t\geq 0} \left\{ \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, x(\gamma(t))\right)}{(t-s)^{1-\zeta}} ds \quad x \in B_r[0] \right\}$$
$$= sup\{ \underbrace{sup}_{t\geq 0} \left\{ \frac{1}{\Gamma(\zeta)} \int_0^t \frac{m_1(s)}{(t-s)^{1-\zeta}} ds \quad x \in B_r[0] \right\}$$
$$\leq \underbrace{sup}_{t\geq 0} \left\{ \underbrace{sup}_{t\geq 0} \frac{v(t)}{\Gamma(\zeta)} \right\} \leq K_1$$
we have $LM = (LK_1) < 1$
Now applying theorem gives that operator $\mathrm{T}(x, y) = \mathcal{A}x.\mathcal{B}y$ has at least a coupled fixed point, which implies (2.1) has a solution on$\mathcal{R}_+$.

Step IV: Now for the local attractivity of the solutions for (2.1), let's assume that *x* and *y* be any two solutions of the (2.1) in $B_r[0]$ defined on$\mathcal{R}_+$. Then we have,

$$|x(t) - y(t)| = \left| [q(t) + f(t, x(\mu(t))] \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, x(\gamma(t))\right)}{(t-s)^{1-\zeta}} ds \right.$$
$$\left. - [q(t) + f(t, x(\mu(t))] \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, y(\gamma(t))\right)}{(t-s)^{1-\zeta}} ds \right|$$
$$|x(t) - y(t)| \leq \left| f(t, x(\mu(t)) \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, x(\gamma(t))\right)}{(t-s)^{1-\zeta}} ds \right| +$$
$$\left| f(t, y(\mu(t)) \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, y(\gamma(t))\right)}{(t-s)^{1-\zeta}} ds \right|$$
$$|x(t) - y(t)| \leq \left| f(t, x(\mu(t)) \right| \left| \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, x(\gamma(t))\right)}{(t-s)^{1-\zeta}} ds \right| +$$
$$\left| f(t, y(\mu(t)) \right| \left| \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, y(\gamma(t))\right)}{(t-s)^{1-\zeta}} ds \right|$$
$$|x(t) - y(t)| \leq \left| f(t, x(\mu(t)) \right| \frac{1}{\Gamma(\zeta)} \int_0^t \frac{\left| g\left(t, x(\gamma(t))\right) \right|}{(t-s)^{1-\zeta}} ds +$$
$$\left| f(t, y(\mu(t)) \right| \frac{1}{\Gamma(\zeta)} \int_0^t \frac{\left| g\left(t, y(\gamma(t))\right) \right|}{(t-s)^{1-\zeta}} ds$$

$$|x(t) - y(t)| \leq |F| \frac{1}{\Gamma(\zeta)} \int_0^t \frac{m_1(s)}{(t-s)^{1-\zeta}} ds + |F| \frac{1}{\Gamma(\zeta)} \int_0^t \frac{m_1(s)}{(t-s)^{1-\zeta}} ds$$

$$|x(t) - y(t)| \leq 2\|F\| \frac{v(t)}{\Gamma(\zeta)}$$

For all $t \in \mathcal{R}_+$. Since and $\lim_{t\to\infty} v(t) = 0$ this gives that $\lim_{t\to\infty} \sup|x(t) - y(t)| = 0$. Thus the (2 .1) has a solution and all the solutions are locally attractive on $\mathcal{R}_+$

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