# EXISTENCE THE SOLUTION FOR FRACTIONAL ORDER QUADRATIC FUNCTIONAL INTEGRAL EQUATION



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**ABSTRACT:An algebraic fixed point theorem involving two operators in Banach Algebras is used to discuss the existence the solution for fractional order quadratic functional integral equation in$\mathcal{R}_+$. Krasonoselkii's fixed point theorem is used here to establish the existence results .Also we prove the existence of maximal and minimal solutions for considered equation. One counter example is considered.**

**KEYWORDS:**Fixed point theorem, Banach algebras, Quadratic functional integral equations, existence result.

## I. INTRODUCTION:

It is well known that integral equations have many useful applications in describing numerous events and problems of real world. The theory of integral equations is rapidly developing using the tools of functional analysis, topology and fixed point theory. Nonlinearquadratic functional-integral equations have been studied in the vehicular traffic, the biology, theory of optimal control and economics, etc. (Argyros, I.K., 1985). There are various cases of functional integral in literature, (Argyros, I.K., 1985; Deimling K., 1985; Banas J., B. Rzepka, 2003; XiaolingHu,Jurang Yan, 2006; Dhage B.C., 2006; Maleknejad K., 2008) and etc.

Nonlinear quadratic integral equations appear very often, in many applications of real world problem. For examples, quadratic integral equations are often applicable in the theory of radiactive transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. Numerous research papers and monographs devoted to differential and integral equations of fractional order have appeared (see [1-10, 17]). In the last 40 year or so, many authors have studied the existence of solutions for several classesof nonlinear quadratic integral equations. For example, Bana´s and Rzepka [21], Caballero *et al.* [18-19], Darwish [25-30] these papers contain various types of existence results for equations of fractional order.

In this paper, we study the existence results of the following nonlinear quadratic functional integral equation of fractional order.

$$x(t) = f\left(t, x(\varphi_1(t))\right) + \left[\frac{1}{\Gamma(\zeta)}\int_0^t \frac{[g(s,x(\varphi_2(s))) + h(s,x(\varphi_3(s)))]}{(t-s)^{1-\zeta}} ds\right] \quad (1.1)$$

for all $t \in \mathcal{R}_+$

Where $f(t,x) = f:\mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ ,$g(t,x) = g:\mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ $h(t,x) = h:\mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$and $\varphi_1 , \varphi_2 , \varphi_3 : \mathcal{R}_+ \to \mathcal{R}_+$

By a solution of the QFIE (1.1) we mean a function $x \in BC(\mathcal{R}_+, \mathcal{R})$that satisfies (1.1) on$\mathcal{R}_+$

Where$BC(\mathcal{R}_+,\mathcal{R})$ is the space of continuous and bounded real-valued functions defined on $\mathcal{R}_+$.In this paper, we prove the existence of the solution for QFIE (1.1) employing Krasonoselkii's fixed point theorem. In the next section, we collect some preliminary definitions and auxiliary results that will be used in this paper.



## II. PRELIMINARIES:

Let $X = BC(\mathcal{R}_+,\mathcal{R})$be Banach algebra with norm $\|\cdot\|$ and let $\Omega$ be a subset of X. Let a mapping $\mathcal{A}: X \to X$be an operator and consider the following operator equation in $X$, namely,$x(t) = (\mathcal{A}x)(t)$(2.1)

Below we give different characterizations of the solutions for operator equation (2.1) on$\mathcal{R}_+$.

***Definition 2.1[20]***:Let $X$ be a Banach space. A mapping $\mathcal{A} : X \to X$ is called Lipschitz if there is a constant$\alpha > 0$ such that $\|\mathcal{A}x - \mathcal{A}y\| \le \alpha\|x - y\|$ for all$x, y \in X$If $\alpha < 1$ then $\mathcal{A}$is called a contractionon $X$ with the contraction constant$\alpha$.

***Definition 2.2***:(Dugundji and Grana's [13]). An operator $\mathcal{A}$ on a Banach space $X$ into itself is called Compact if for any bounded subset S of $X$, $\mathcal{A}$(S) is a relatively compact subset of $X$. If $\mathcal{A}$ is continuous and compact, then it is called completely continuous on $X$.

Let $X$ be a Banach space with the norm $\|.\|$and Let$\mathcal{A} : X \to X$ be an operator (in general nonlinear). Then $\mathcal{A}$ is called

(i) Compact if $\mathcal{A}(X)$ is relatively compact subset of $X$;

(ii) Totally bounded if $\mathcal{A}(S)$ is a totally bounded subset of $X$ for any bounded subset S of $X$

(iii) Completely continuous if it is continuous and totally bounded operator on $X$.

It is clear that every compact operator is totally bounded but the converse need not be true.

The solutions of (1.1) in the space $BC(\mathcal{R}_+,\mathcal{R})$of continuous and bounded real-valued functions defined on$\mathcal{R}_+$. Define a standard supremum norm $\|.\|$ and a multiplication "." in $BC(\mathcal{R}_+,\mathcal{R})$by

$\|x\| = \sup\{|x(t)|: t \in \mathcal{R}_+\}$(2.2)

$(xy)(t) = x(t)y(t)t \in \mathcal{R}_+$(2.3)

***Definition 2.4[17]***:The Riemann-Liouville fractional integral of order $\beta$of the function $x(t)$ is defined by the formula:

$$I^{\beta} x(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{x(s)}{(t-s)^{1-\beta}} ds \ \ t \in (a,b) \tag{2.4}$$

Where $\Gamma(\beta)$ denote the gamma function.

It may be shown that the fractional integral operator $I^{\beta}$ transforms the space $\mathcal{L}^1(a,b)$ into itself and has some other properties (see [12-19])

***Definition 2.5[17]***:The left sided Riemann-Liouville fractional integral [10, 12, 18] of order $\beta$ of real function f is defined as

$$I^{\beta}_{a^+} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{f(t)}{(t-s)^{1-\beta}} dt \ \ \beta > 0, x > a \tag{2.5}$$

***Theorem 2.1:***(Arzela-Ascoli theorem) (8): If every uniformly bounded and equi-continuous sequence $\{f_n\}$ of functions in$C(\mathcal{J},\mathcal{R})$, then it has a convergent subsequence.

***Theorem 2.2[8]:*** A metric space $X$ is compact iff every sequence in $X$ has a convergent subsequence.



**III. EXISTENCE RESULTS:**

***Definition 3.1:*** A mapping $g:\mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is said to be Caratheodory if

1. $t \to g(t,x)$ is measurable for all $x \in \mathcal{R}$, and
2. $x \to g(t,x)$ is continuous almost everywhere for $t \in \mathcal{R}_+$

   Again a caratheodory function $g$ is called $\mathcal{L}^1$-Caratheodory if

3. for each real number $r > 0$ there exists a function $h_r \in \mathcal{L}^1(\mathcal{R}_+,\mathcal{R})$ such that $|g(t,x)| \le h_r(t)$ a.e. $t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$ with $|x| \le r$

Finally, a Caratheodory function $g(t,x)$ is called $\mathcal{L}^1_{\mathcal{R}}$ − Caratheodory if

4. there exist a function $h \in \mathcal{L}^1(\mathcal{R}_+,\mathcal{R})$ such that $|g(t,x)| \le h(t)$

   a.e. $t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$

For convenience, the function $h$ is referred to as a bound function of $g$.

We recall the fixed point theorem due to Krasonoselkii's.

***Theorem 3.1 :( Krasonoselkii's) (31, 32, and 40)*** *Let X be a Banach Space and D be a bounded closed convex subset of X. Let $\mathcal{A}$, $\mathcal{B}$ maps D into X s.t. $\mathcal{A}u + \mathcal{B}u \in D$ for every (u, v) ∈ D.*

*If $\mathcal{A}$ is a contraction and $\mathcal{B}$ is completely continuous then the equation $\mathcal{A}w + \mathcal{B}w = w$ has a solution w on D. i.e.*

*a) $\mathcal{A}$ is a contraction*

*b) $\mathcal{B}$ is completely continuous*

*c) $\mathcal{A}u + Bu \in D$.*

***Hypothesis:***

Assume that following hypotheses are satisfied.

a) The function $\varphi_1, \varphi_2, \varphi_3 : \mathcal{R}_+ \to \mathcal{R}_+$ are continuous.

b) The function $g, h: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is continuous and

Satisfying $|g(t,x) - g(t,y)| \le \varphi_2(t)|x-y|$ $t \in \mathcal{R}_+$ for all $x, y \in \mathcal{R}$

$|h(t,x) - h(t,y)| \le \varphi_3(t)|x-y|$ $t \in \mathcal{R}_+$ for all $x, y \in \mathcal{R}$

c) The function $f:\mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is continuous and bounded with bound

$F = \underbrace{sup}_{(t,x)\in\mathcal{R}_+\times\mathcal{R}} |f(t,x)|$ there exists a bounded function $l : \mathcal{R}_+ \to \mathcal{R}_+$ with bound L satisfying

$|f(t,x) - f(t,y)| \le \frac{l(t)|x-y|}{2(N+|x-y|)}$ $t \in \mathcal{R}_+$, for all $x, y \in \mathcal{R}$ and $0 < L \le N$ and vanishes as $\lim_{t\to\infty}$

d) The function $g, h: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ satisfy caratheodory condition (i.e.measrable in $t$ for all $x \in \mathcal{R}$ and continuous in $x$ for all $t \in \mathcal{R}_+$) and there exist function $m \in \mathcal{L}^1(\mathcal{R}_+,\mathcal{R})$ Such that $g(t,x) + h(t,x) \le m(t) \forall (t,x) \in \mathcal{R}_+ \times \mathcal{R}$ where $t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$ and $\lim_{t\to\infty} \int_0^t m(s)ds = 0$

e)The uniform continuous function $v: \mathcal{R}_+ \to \mathcal{R}_+$ defined by the formulas

$$v(t) = \int_0^t \frac{m(s)}{(t-s)^{1-\zeta}} ds$$

is bounded on $\mathcal{R}_+$ and vanish at infinity, that is, $\lim_{t\to\infty} v(t) = 0$.



***Remark 3.1:*** Note that if the hypothesis (b) and (d) hold, then there exist constant $K > 0$ such that:

$$K = sup \frac{1}{\Gamma(\zeta)} \int_0^t \frac{m(s)}{(t-s)^{1-\zeta}} ds \quad and \quad \int_0^t \frac{[\|\varphi_2\| + \|\varphi_3\|]}{(t-s)^{1-\zeta}} ds \le \Gamma(\zeta)$$

***Theorem 3.2 :*** Suppose that the hypothesis [(a) – (e)] holds. Then the QFIE (6.4.1) has a solution in the space $BC(\mathcal{R}_+, \mathcal{R})$

Proof: Consider the problem

$$x(t) = \left[\frac{1}{\Gamma(\zeta)} \int_0^t \frac{[g(s, x(\varphi_2(s))) + h(s, x(\varphi_3(s)))]}{(t-s)^{1-\zeta}} ds\right] + f\left(t, x(\varphi_1(t))\right)$$

For all $t \in \mathcal{R}_+$

Where $f(t,x) = f: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}, g(t,x) = g: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}, h: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ and $\varphi_1, \varphi_2, \varphi_3 : \mathcal{R}_+ \to \mathcal{R}_+$

Now we define two operators $\mathcal{A}, \mathcal{B} : D \to X$ *s.t.*

$$\mathcal{A}x(t) = f\left(t, x(\varphi_1(t))\right)$$

$$\mathcal{B}x(t) = \left[\frac{1}{\Gamma(\zeta)} \int_0^t \frac{[g(s, x(\varphi_2(s))) + h(s, x(\varphi_3(s)))]}{(t-s)^{1-\zeta}} ds\right]$$

Consider the closed ball $B_r[0]=D$ in $X$ centered at origin 0 and of radius $r$, where

$$r = F + (K) > 0$$

We show that $\mathcal{A}$ and $\mathcal{B}$ satisfy all the requirements of theorem 3.1 on $D$

***STEP I] $\mathcal{A}$ is Contraction***

Let $x, y \in X$ be arbitrary, and then by hypothesis (b) and (c), we get

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = \left|f\left(t, x(\varphi_1(t))\right) - f\left(t, y(\varphi_1(t))\right)\right|$$

$$\le \frac{l(t)\left|x(\varphi_1(t)) - y(\varphi_1(t))\right|}{2(N + |x-y|)}$$

$$\le \frac{L\left|x(\varphi_1(t)) - y(\varphi_1(t))\right|}{2(N + |x-y|)} \text{for all } \quad t \in \mathcal{R}_+$$

Taking supremum over t

$$\|\mathcal{A}x - \mathcal{A}y\| \le \frac{L\|x-y\|}{2(N + \|x-y\|)} \quad \text{for all } x, y \in X$$

This shows that $\mathcal{A}$ is contraction mapping $L_1 = \frac{L}{2(N+\|x-y\|)}$

***Step II] To show that $\mathcal{B}$ is continuous and compact operator***

Firstwe show that $\mathcal{B}$ is continuous on D

CaseI:Suppose that $t \geq T$ there exist $T > 0$ and let us fix arbitrary $\varepsilon > 0$ and take $x,y \in D$ such that $\|x-y\| \leq \varepsilon$. Then

$|(\mathcal{B}x)t - (\mathcal{B}y)t|$

$$\leq \left|\frac{1}{\Gamma(\zeta)}\int_0^t \frac{[g(s,x(\varphi_2(s))) + h(s,x(\varphi_3(s)))]}{(t-s)^{1-\zeta}}ds - \frac{1}{\Gamma(\zeta)}\int_0^t \frac{[g(s,y(\varphi_2(s))) + h(s,y(\varphi_3(s)))]}{(t-s)^{1-\zeta}}ds\right|$$

$$\leq \left|\frac{1}{\Gamma(\zeta)}\int_0^t \frac{g\left(s,x(\varphi_2(s))\right) - g\left(s,y(\varphi_2(s))\right)}{(t-s)^{1-\zeta}}ds - \frac{1}{\Gamma(\zeta)}\int_0^t \frac{h\left(s,x(\varphi_3(s))\right) - h\left(s,y(\varphi_3(s))\right)}{(t-s)^{1-\zeta}}ds\right|$$

$$\leq \frac{1}{\Gamma(\zeta)}\left[\int_0^t \frac{\left|g\left(s,x(\varphi_2(s))\right) - g\left(s,y(\varphi_2(s))\right)\right|}{(t-s)^{1-\zeta}}ds + \int_0^t \frac{\left|h\left(s,x(\varphi_3(s))\right) - h\left(s,y(\varphi_3(s))\right)\right|}{(t-s)^{1-\zeta}}ds\right]$$

$$\leq \frac{1}{\Gamma(\zeta)}\left[\int_0^t \frac{\|\varphi_2\|\|x-y\|}{(t-s)^{1-\zeta}}ds + \int_0^t \frac{\|\varphi_3\|\|x-y\|}{(t-s)^{1-\zeta}}ds\right]$$

$$\leq \frac{\|x-y\|}{\Gamma(\zeta)}\left[\int_0^t \frac{[\|\varphi_2\| + \|\varphi_3\|]}{(t-s)^{1-\zeta}}ds\right]$$

$$\leq \frac{\varepsilon}{\Gamma(\zeta)}\Gamma(\zeta)$$

for all $t \geq T$

Since $\varepsilon$ is an arbitrary, we derive that

$|(\mathcal{B}x)t - (\mathcal{B}y)t| \leq \varepsilon$

CaseII: Further, let us assume that $t \in [0,T]$, then

$|(\mathcal{B}x)t - (\mathcal{B}y)t|$

$$\leq \left|\frac{1}{\Gamma(\zeta)}\int_0^t \frac{[g(s,x(\varphi_2(s))) + h(s,x(\varphi_3(s)))]}{(t-s)^{1-\zeta}}ds - \frac{1}{\Gamma(\zeta)}\int_0^t \frac{[g(s,y(\varphi_2(s))) + h(s,y(\varphi_3(s)))]}{(t-s)^{1-\zeta}}ds\right|$$

$$\leq \left|\frac{1}{\Gamma(\zeta)}\int_0^T \frac{[g(s,x(\varphi_2(s))) + h(s,x(\varphi_3(s)))]}{(t-s)^{1-\zeta}}ds - \frac{1}{\Gamma(\zeta)}\int_0^T \frac{[g(s,y(\varphi_2(s))) + h(s,y(\varphi_3(s)))]}{(t-s)^{1-\zeta}}ds\right|$$

$$\leq \frac{1}{\Gamma(\zeta)}\left[\int_0^T \frac{\left|g\left(s,x(\varphi_2(s))\right) - g(s,y(\varphi_2(s)))\right| + \left|h\left(s,x(\varphi_3(s))\right) - h(s,y(\varphi_3(s)))\right|}{(t-s)^{1-\zeta}}ds\right]$$

$$\leq \frac{1}{\Gamma(\zeta)}\left[\int_0^T \frac{w_r^T(g,\epsilon) + w_r^T(h,\epsilon)}{(t-s)^{\zeta}}ds\right]$$

$$\leq \frac{1}{\Gamma(\zeta)}\left[\frac{w_r^T(g,\epsilon) + w_r^T(h,\epsilon)}{\zeta}T^{\zeta}ds\right]$$

$$\leq \left[\frac{w_r^T(g,\epsilon)+w_r^T(h,\epsilon)}{\Gamma(\zeta+1)}T^{\zeta}ds\right]$$

Where

$w_r^T(g,\epsilon)=\sup\{|g(s,x)-g(s,y)|: s\in[0,T]; x,y\in[-r,r], |x-y|\leq\epsilon\}$

$w_r^T(h,\epsilon)=\sup\{|h(s,x)-h(s,y)|: s\in[0,T]; x,y\in[-r,r], |x-y|\leq\epsilon\}$

Therefore, from the uniform continuity of the function $g(t,x) and\ h(t,x)$ on the set $[0,T]\times[-r,r]$; .we derive that $w_r^T(g,\epsilon)+w_r^T(h,\epsilon)\to 0$ as $\epsilon\to 0$.

Now from case I and II, we conclude that the operator $\mathcal{B}$ is continuous operator on closed ball$D$ into itself.

***Step III] Next we show that $\mathcal{B}$ is compact on D***

(A) First prove that every sequence $\{\mathcal{B}x_n\}$ inD has a uniformly bounded sequence in$D$.Now by(c) – (e)

$$|(\mathcal{B}x_n)t| = \left|\frac{1}{\Gamma(\zeta)}\int_0^t \frac{g(s,x_n(\varphi_2(s)))+h(s,x_n(\varphi_3(s)))}{(t-s)^{1-\zeta}}ds\right|$$

$$|(\mathcal{B}x_n)t| \leq \frac{1}{\Gamma(\zeta)}\int_0^t \frac{|g(s,x_n(\varphi_2(s)))+h(s,x_n(\varphi_3(s)))|}{(t-s)^{1-\zeta}}ds$$

$$|(\mathcal{B}x_n)t| \leq \frac{1}{\Gamma(\zeta)}\int_0^t \frac{m(s)}{(t-s)^{1-\zeta}}ds$$

$$|(\mathcal{B}x_n)t| \leq \frac{v(t)}{\Gamma(\zeta)}$$

$$|(\mathcal{B}x_n)t| \leq K \ \ \forall t\in R_+$$

Taking supremum over t, we obtain $\|\mathcal{B}x_n\|\leq K \quad \forall n\in N$

This shows that $\{\mathcal{B}x_n\}$ is a uniformly bounded sequence in$D$.

(B) Now to that show the sequence $\{\mathcal{B}x_n\}$is equicontinuous sequence. Let$\varepsilon>0$ be given.

i) If $t_1$ , $t_2\in[0,T]$ then we have

$|(\mathcal{B}x_n)t_2-(\mathcal{B}x_n)t_1|$

$$\leq\left|\frac{1}{\Gamma(\zeta)}\int_0^{t_2}\frac{g(s,x_n(\varphi_2(s)))+h(s,x_n(\varphi_3(s)))}{(t_2-s)^{1-\zeta}}ds-\frac{1}{\Gamma(\zeta)}\int_0^{t_1}\frac{g(s,x_n(\varphi_2(s)))+h(s,x_n(\varphi_3(s)))}{(t_1-s)^{1-\zeta}}ds\right|$$

$$\leq\left|\frac{1}{\Gamma(\zeta)}\int_0^{t_2}\frac{|g(s,x_n(\varphi_2(s)))+h(s,x_n(\varphi_3(s)))|}{(t_2-s)^{1-\zeta}}ds-\frac{1}{\Gamma(\zeta)}\int_0^{t_1}\frac{|g(s,x_n(\varphi_2(s)))+h(s,x_n(\varphi_3(s)))|}{(t_1-s)^{1-\zeta}}ds\right|$$

$$\leq\left|\frac{1}{\Gamma(\zeta)}\int_0^{t_2}\frac{m(s)}{(t_2-s)^{1-\zeta}}ds-\frac{1}{\Gamma(\zeta)}\int_0^{t_1}\frac{m(s)}{(t_1-s)^{1-\zeta}}ds\right|$$

$$\leq\frac{1}{\Gamma(\zeta)}\left|\int_0^{t_2}\frac{m(s)}{(t_2-s)^{1-\zeta}}ds-\int_0^{t_1}\frac{m(s)}{(t_1-s)^{1-\zeta}}ds\right|$$

$$\leq\frac{1}{\Gamma(\zeta)}|v(t_2)-v(t_1)|$$

By the uniform continuity of the function $v(t)$ on $[0,T]$, we get

$|(\mathcal{B}x_n)t_2-(\mathcal{B}x_n)t_1|\to 0$ as $t_1\to t_2$

ii) If $t_1$ , $t_2 \geq T$ then we have

$|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1|$

$$\leq \left|\frac{1}{\Gamma(\zeta)}\int_0^{t_2} \frac{g(s,x_n(\varphi_2(s)) + h(s,x_n(\varphi_3(s))}{(t_2 - s)^{1-\zeta}} ds - \frac{1}{\Gamma(\zeta)}\int_0^{t_1} \frac{g(s,x_n(\varphi_2(s)) + h(s,x_n(\varphi_3(s))}{(t_1 - s)^{1-\zeta}} ds\right|$$

$$\leq \left|\frac{1}{\Gamma(\zeta)}\int_0^{t_2} \frac{g(s,x_n(\varphi_2(s))) + h(s,x_n(\varphi_3(s)))}{(t_2 - s)^{1-\zeta}} ds - \frac{1}{\Gamma(\zeta)}\int_0^{t_1} \frac{g(s,x_n(\varphi_2(s))) + h(s,x_n(\varphi_3(s)))}{(t_1 - s)^{1-\zeta}} ds\right|$$

$$\leq \left|\frac{1}{\Gamma(\zeta)}\int_0^{t_2} \frac{g(s,x_n(\varphi_2(s))) + h(s,x_n(\varphi_3(s)))}{(t_2 - s)^{1-\zeta}} ds\right| + \left|\frac{1}{\Gamma(\zeta)}\int_0^{t_1} \frac{g(s,x_n(\varphi_2(s))) + h(s,x_n(\varphi_3(s)))}{(t_1 - s)^{1-\zeta}} ds\right|$$

$$\leq \frac{v(t_2)}{\Gamma(\zeta)} + \frac{v(t_1)}{\Gamma(\zeta)} \leq 0 + \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$\leq \epsilon$ As $t_1 \to t_2$.

iii) If$t_1$ , $t_2 \in \mathcal{R}_+$

With $t_1 < \mathrm{T} < t_2$ then we have

$|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1| \leq |\mathcal{B}x_n(t_2) - \mathcal{B}x_n(T)| + |\mathcal{B}x_n(T) - \mathcal{B}x_n(t_1)|$

Now if $t_1 \to t_2$ $then t_1 \to$T and T$\to t_2$

Therefore,$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(T)| \to 0$ $and$ $|\mathcal{B}x_n(T) - \mathcal{B}x_n(t_1)| \to 0$

And so $|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1| \to 0$ $as t_1 \to t_2$ for all$t_1$ , $t_2 \in \mathcal{R}_+$

Hence $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in X. So by the Arzela-Ascoli theorem $\{\mathcal{B}x_n\}$has a uniformly convergent subsequence on the compact subset $[0,T]$of $\mathcal{R}_+$. We call the subsequence of the sequence itself. This Yields that $\mathcal{B}$ is compact on$D$.

So $\mathcal{B}$is completely continuous.

***Step IV] Next we show that $\mathcal{A}x + \mathcal{B}x \in D$***

For all$x, y \in D$ are arbitrary, then

$|\mathcal{A}x(t) + \mathcal{B}x(t)| \leq |\mathcal{A}x(t)| + |\mathcal{B}x(t)|$

$$|\mathcal{A}x(t) + \mathcal{B}x(t)| \leq \left\|\left[\frac{1}{\Gamma(\zeta)}\int_0^{t} \frac{g(s,x(\varphi_2(s))) + h(s,x(\varphi_3(s)))}{(t - s)^{1-\zeta}} ds\right]\right\| + \left|f\left(t,x(\varphi_1(t))\right)\right|$$

$$\leq \left[\frac{1}{\Gamma(\zeta)}\int_0^{t} \frac{|g(s,x(\varphi_2(s))) + h(s,x(\varphi_3(s)))|}{(t - s)^{1-\zeta}} ds\right] + F$$

$$\leq F + \left[\frac{1}{\Gamma(\zeta)}\int_0^{t} \frac{m(s)}{(t - s)^{1-\zeta}} ds\right]$$

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$$\leq F + \left[\frac{v(t)}{\Gamma(\zeta)}\right]$$

$$\leq F + [K] = r \quad \text{for all t in } \mathcal{R}_+$$

Taking the supremum over t, we obtain $\| \mathcal{A}x + \mathcal{B}x \| \leq r$ for all$x \in D$

Hence hypothesis (c) of Theorem holds. Nowapplyingtheorem [3.1] gives that QFIE (1.1) has a solution on $\mathcal{R}_+$.

Step V:Now for the local attractivity of the solutions for (1.1), let's assume that $x$ and $y$ be any two solutions of the (1.1) in $D$ defined on$\mathcal{R}_+$ . Then we have,

$$|x(t) - y(t)| = \left| f\left(t, x(\varphi_1(t))\right) + \left[\frac{1}{\Gamma(\zeta)}\int_0^t \frac{\left[g\left(s, x(\varphi_2(s))\right) + h\left(s, x(\varphi_3(s))\right)\right]}{(t-s)^{1-\zeta}} ds\right] - f\left(t, y(\varphi_1(t))\right) - \left[\frac{1}{\Gamma(\zeta)}\int_0^t \frac{\left[g\left(s, y(\varphi_2(s))\right) + h\left(s, y(\varphi_3(s))\right)\right]}{(t-s)^{1-\zeta}} ds\right]\right|$$

$$|x(t) - y(t)| \leq 2F + \left\|\left[\frac{1}{\Gamma(\zeta)}\int_0^t \frac{\left[g\left(s, x(\varphi_2(s))\right) + h\left(s, x(\varphi_3(s))\right)\right]}{(t-s)^{1-\zeta}} ds\right]\right\| + \left\|\left[\frac{1}{\Gamma(\zeta)}\int_0^t \frac{\left[g\left(s, y(\varphi_2(s))\right) + h\left(s, y(\varphi_3(s))\right)\right]}{(t-s)^{1-\zeta}} ds\right]\right\|$$

$$|x(t) - y(t)| \leq 2F + \left[\frac{1}{\Gamma(\zeta)}\int_0^t \frac{\left|g\left(s, x(\varphi_2(s))\right) + h\left(s, x(\varphi_3(s))\right)\right|}{(t-s)^{1-\zeta}} ds\right] + \left[\frac{1}{\Gamma(\zeta)}\int_0^t \frac{\left|g\left(s, y(\varphi_2(s))\right) + h\left(s, y(\varphi_3(s))\right)\right|}{(t-s)^{1-\zeta}} ds\right]$$

$$|x(t) - y(t)| \leq 2F + \left[\frac{1}{\Gamma(\zeta)}\int_0^t \frac{m(s)}{(t-s)^{1-\zeta}} ds\right] + \left[\frac{1}{\Gamma(\zeta)}\int_0^t \frac{m(s)}{(t-s)^{1-\zeta}} ds\right]$$

$$|x(t) - y(t)| \leq 2F + 2\left[\frac{v(t)}{\Gamma(\zeta)}\right]$$

For all$t \in \mathcal{R}_+$ as $\lim_{t\to\infty} v(t) = 0$ this gives that $\lim_{t\to\infty} \sup|x(t) - y(t)| = 0$. Thus (1.1) has a solution and all the solutions are locally attractive on$\mathcal{R}_+$.

## IV. EXISTENCE OF EXTREMAL SOLUTIONS:

***Definition 4.1:*** ***(Caratheodory case)***A function $\tau: \mathcal{R} \to \mathcal{R}$is nondecreasing if $\tau(x) \leq \tau(y) \forall x, y \in \mathcal{R}$ for which $x \leq y$. similarly $\tau(x)$ is increasing in $x$ if $\tau(x) < \tau(y) \forall x, y \in \mathcal{R}$ for which $x < y$.

***Definition 4.2:*** A function $p_1 \in BC(\mathcal{R}_+, \mathcal{R})$ is called a lower solution of the QFIE (1.1) on $\mathcal{R}_+$ if the function $t \to \left\{p_1(t) - f\left(t, p_1(\varphi_1(t))\right)\right\}$ is continuous absolutely and

$$p_1(t) \leq \left[\frac{1}{\Gamma(\zeta)}\int_0^t \frac{g(s, p_1(\varphi_2(s))) + h(s, p_1(\varphi_3(s)))}{(t-s)^{1-\zeta}} ds\right] + f\left(t, p_1(\varphi_1(t))\right) \quad (4.1)$$

Again a function $p_2 \in BC(\mathcal{R}_+, \mathcal{R})$ is called an upper solution of the QFIE (1.1) on$\mathcal{R}_+$ if the function $t \to \left\{p_1(t) - f\left(t, p_2(\varphi_1(t))\right)\right\}$ is continuous absolutely and

$$p_2(t) \geq \left[\frac{1}{\Gamma(\zeta)}\int_0^t \frac{g(s, p_2(\varphi_2(s))) + h(s, p_2(\varphi_3(s)))}{(t-s)^{1-\zeta}} ds\right] + f\left(t, p_2(\varphi_1(t))\right) \quad (4.2)$$

***Definition 4.3:*** A solution $x_M$ of the QFIE (1.1) is said to be maximal if for any other solution $x$ to QFIE (1.1) one has $x(t) \leq x_M(t)$ for all $t \in \mathcal{R}_+$.

Again a solution $x_M$ of the QFIE (1.1) is said to be minimal if $x_M(t) \leq x(t)$ for all $t \in \mathcal{R}_+$ where $x$ is any solution of the QFIE (1.1) on $\mathcal{R}_+$.

***Definition 4.4[10, 41]:*** A closed and non-empty set $\mathcal{K}$ in a Banach Algebra $X$ is called a cone if

i. $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$

ii. $\lambda\mathcal{K} \subseteq \mathcal{K}$ for $\lambda \in \mathcal{K}, \lambda \geq 0$

iii. $\{-\mathcal{K}\} \cap \mathcal{K} = 0$ Where 0 is the zero element of $X$.

and is called positive cone if

iv. $\mathcal{K} \circ \mathcal{K} \subseteq \mathcal{K}$

And the notation $\circ$ is a multiplication composition in $X$

We introduce an order relation $\leq$ in $X$ as follows.

Let $x, y \in X$ then $x \leq y$ if and only if $y - x \in \mathcal{K}$. A cone $\mathcal{K}$ is called normal if the norm $\|\cdot\|$ is monotone increasing on $\mathcal{K}$. It is known that if the cone $\mathcal{K}$ is normal in $X$ then every order-bounded set in $X$ is norm-bounded set in $X$. The details of cone and their properties appear in Guo and Lakshikantham [35, 36]

We equip the space $C(\mathcal{R}_+, \mathcal{R})$ of continuous real valued function on $\mathcal{R}_+$ with the order relation $\leq$ with the help of cone defined by,

$\mathcal{K} = \{x \in C(\mathcal{R}_+, \mathcal{R}): x(t) \geq 0\ \forall t \in \mathcal{R}_+\}$ (4.4.1)

We well known that the cone $\mathcal{K}$ is normal and positive in $C(\mathcal{R}_+, \mathcal{R})$. As a result of positivity of the cone $\mathcal{K}$ we have:

***Lemma 4.5[06]:*** Let $p_1, p_2, q_1, q_2 \in \mathcal{K}$ be such that $p_1 \leq q_1$ and $p_2 \leq q_2$ then $p_1 p_2 \leq q_1 q_2$.

For any $p_1, p_2 \in X = C(\mathcal{R}_+, \mathcal{R}), p_1 \leq p_2$ the order interval $[p_1, p_2]$ is a set in $X$ given by, $[p_1, p_2] = \{x \in X: p_1 \leq x \leq p_2\}$

***Definition 4.6[06]:*** A mapping $R: [p_1, p_2] \to X$ is said to be nondecreasing or monotone increasing if $x \leq y$ implies $Rx \leq Ry$ for all $x, y \in [p_1, p_2]$.

***Theorem 4.7[12]:*** Let $\mathcal{K}$ be a cone in Banach Algebra $X$ and let $[p_1, p_2] \in X$. Suppose that $\mathcal{A}, \mathcal{B}: [p_1, p_2] \to \mathcal{K}$ be two operators such that

a. $\mathcal{A}$ is Lipschitz with Lipschitz constant $\alpha$

b. $\mathcal{B}$ is completely continuous,

c. $\mathcal{A}x + \mathcal{B}x \in [p, q]$ for each $x \in [p, q]$ and

d. $\mathcal{A}$ and $\mathcal{B}$ are nondecreasing.

Further if the cone $\mathcal{K}$ is normal and positive then the operator equation $x = \mathcal{A}x + \mathcal{B}x$ has the least and greatest positive solution in $[p_1, p_2]$ whenever $\alpha M < 1$, where $M = \|\mathcal{B}([p_1, p_2])\| = sup\{\|\mathcal{B}x\|: x \in [p_1, p_2]\}$

We assume the following hypothesis

*i)* The function $g$ *and* $h$ are caratheodory.

*ii)* The function $x \to \left\{x(t) - f\left(t, x(\varphi_1(t))\right)\right\}$ is increasing in the interval $\left[min_{t \in \mathcal{R}_+} p_1(t), max_{t \in \mathcal{R}_+} p_2(t)\right]$.

*iii)* The functions $g, h: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$, $f: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ are nondecreasing in $x$ almost everywhere for $t \in \mathcal{R}_+$.

*iv)* The QFIE (1.1) has a lower solution $p_1$ and upper solution $p_2$ on $\mathcal{R}_+$ with $p_1 \leq p_2$.

*v)* The function $m: \mathcal{R}_+ \to \mathcal{R}$ defined by

$m(t) = \left|g(s, p_1(\varphi_2(s))) + h(s, p_1(\varphi_3(s)))\right| + \left|g(s, p_2(\varphi_2(s))) + h(s, p_2(\varphi_3(s)))\right|$ is Lebesgue measurable.

***Remark 4.4:*** Assume that the hypotheses $(i - v)$ holds, then

$|g(s, x(\varphi_2(s))) + h(s, x(\varphi_3(s)))| \leq m(t)$, a.e. $t \in \mathcal{R}_+$

***Theorem 4.5:*** The QFIE (1.1) has minimal and maximal positive solution on $\mathcal{R}_+$. If the hypothesis $(a - e)$ and $(i - v)$ holds and $m$ is given in above remark, further

$$\|L_1\| \left\{ \frac{1}{\Gamma(\zeta + 1)} T^{\zeta} \|m\|_{\mathcal{L}^1} \right\} < 1$$

***Proof:*** Let $X = BC(\mathcal{R}_+, \mathcal{R})$ and we define an order relation "$\leq$" by the cone $\mathcal{K}$ given by (4.4). Clearly $\mathcal{K}$ is a normal cone in $X$. Define two operators $\mathcal{A}$, $\mathcal{B}$ on $X$ as in previous proof. Then QFIE (1.1) is transformed into an operator equation $\mathcal{A}x + \mathcal{B}x = x$ in Banachalgebra $X$. Notice that $(iii)$ implies $\mathcal{A}, \mathcal{B}: [p_1, p_2] \to \mathcal{K}$ also note that $(iv)$ ensures that $p_1 \leq \mathcal{A}p_1 + \mathcal{B}p_1$ and $\mathcal{A}p_2 + \mathcal{B}p_2 \leq p_2$.since the cone $\mathcal{K}$ in $X$ is normal, $[p_1, p_2]$is a norm bounded set in $X$. Now it is shown, as in the proof of Theorem (1.1) that$\mathcal{A}$iscontraction mapping. Also we have shown that$\mathcal{B}$ is completely continuous operator

Now byusing$(iii)$, let $x, y \in [p_1, p_2]$ be such that $x \leq y$

$$\mathcal{A}x(t) = f\left(t, x(\varphi_1(t))\right) \leq f\left(t, y(\varphi_1(t))\right) \leq \mathcal{A}y(t), \forall t \in \mathcal{R}_+$$

and,

$$\mathcal{B}x(t) = \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g(s, x(\varphi_2(s))) + h(s, x(\varphi_3(s)))}{(t - s)^{1-\zeta}} ds$$

$$\leq \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g(s, y(\varphi_2(s))) + h(s, y(\varphi_3(s)))}{(t - s)^{1-\zeta}} ds \leq \mathcal{B}y(t) t \in \mathcal{R}_+$$

From this we conclude that $\mathcal{A}, \mathcal{B}$ are non-decreasing operators on $[p_1, p_2]$.

Again definition (4.2) and hypothesis $(iv)$ implies that

$$p_1(t) \leq \left[ \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g(s, p_1(\varphi_2(s))) + h(s, p_1(\varphi_3(s)))}{(t - s)^{1-\zeta}} ds \right] + f\left(t, p_1(\varphi_1(t))\right)$$

$$\leq \left[ \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g(s, x(\varphi_2(s))) + h(s, x(\varphi_3(s)))}{(t - s)^{1-\zeta}} ds \right] + f\left(t, x(\varphi_1(t))\right)$$

$$\leq \left[ \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g(s, p_2(\varphi_2(s))) + h(s, p_2(\varphi_3(s)))}{(t - s)^{1-\zeta}} ds \right] + f\left(t, p_2(\varphi_1(t))\right)$$

$\leq p_2(t), \forall t \in \mathcal{R}_+$ and $x \in [p_1, p_2]$

As a result $p_1(t) \leq \mathcal{A}x(t) + \mathcal{B}x(t) \leq p_2(t), \forall t \in \mathcal{R}_+$ and $x \in [p_1, p_2]$

Hence $\mathcal{A}x + \mathcal{B}x \in [p_1, p_2] \forall x \in [p_1, p_2]$

Again $M = \|\mathcal{B}([p_1, p_2])\| = sup\{\|\mathcal{B}x\| : x \in [p_1, p_2]\}$

$$\leq sup\left\{ sup_{t \in \mathcal{R}_+} \left\{ \frac{1}{\Gamma(\zeta)} \int_0^t \frac{|g(s, x(\varphi_2(s))) + h(s, x(\varphi_3(s)))|}{(t - s)^{1-\zeta}} ds \right\} : x \in [p_1, p_2] \right\}$$

$$\leq sup\left\{ \frac{1}{\Gamma(\zeta)} \left[ \frac{(t - s)^{\zeta}}{-\zeta} \right]_0^t \|m\|_{\mathcal{L}^1} \right\} \leq \frac{1}{\Gamma(\zeta + 1)} T^{\zeta} \|m\|_{\mathcal{L}^1}$$

Since

$$L_1 M \leq \|L_1\| \left\{ \frac{1}{\Gamma(\zeta + 1)} T^{\zeta} \|m\|_{\mathcal{L}^1} \right\} < 1$$

We apply theorem (4.7) to the operator equation $\mathcal{A}x + \mathcal{B}x = x$ to yield that the QFIE (1.1) has minimum and maximum positive solution on $\mathcal{R}_+$.

## V. EXAMPLE:

Let'sconsiderthe following fractional order QFIE of type (1.1)

$$x(t)=\frac{cost}{t+1}\left[1+\frac{x(t)}{4+x(t)}\right]+\left[\int_0^t\frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)}\left[\frac{t}{t^7(t^2-2+x(8t))}-\frac{t}{t^7(2t+x(t))}\right]\right] \qquad \forall t\in\mathcal{R}_+$$

***Solution***:Here

$$f\left(t,x(\varphi_1(t))\right)=\frac{cost}{t+1}\left[1+\frac{x(t)}{4+x(t)}\right] \text{ and}$$

$$g(s,x(\varphi_2(s))+h(s,x(\varphi_3(s))=\frac{t}{t^7(t^2-2+x(8t))}-\frac{t}{t^7(2t+x(t))} \text{ and } h(t)=\frac{s^2}{t^6}$$

*i)* Here $\zeta=\frac{1}{3}$and $\varphi_1(t)=t,\varphi_2(t)=8t,\varphi_3(t)=t$are continuous.

*ii)* $\left|g\left(t,x(\varphi_2(t))\right)-g\left(t,y(\varphi_2(t))\right)\right|=\left|\frac{t}{t^7(t^2-2+x(8t))}-\frac{t}{t^7(t^2-2+y(8t))}\right|$

$$\leq\left|\frac{t^2-2+y(8t)-t^2+2-x(8t)}{t^6(t^2-2+x(8t))(t^2-2+y(8t))}\right|\leq\left|\frac{y(8t)-x(8t)}{t^6(t^2-2+x(8t))(t^2-2+y(8t))}\right|$$

$$\leq 8t|x(t)-y(t)|$$

$$\left|h\left(t,x(\varphi_3(t))\right)-h\left(t,y(\varphi_3(t))\right)\right|=\left|-\frac{t}{t^7(2t+x(t))}+\frac{t}{t^7(2t+y(t))}\right|$$

$$\leq\left|\frac{2t+x(t)-2t-y(t)}{t^6(t^2-2+x(8t))(t^2-2+y(8t))}\right|\leq\left|\frac{x(t)-y(t)}{t^6(t^2-2+x(8t))(t^2-2+y(8t))}\right|$$

$$\leq t|x(t)-y(t)|$$

*iii)* $\left|f\left(t,x(\varphi_1(t))\right)-f\left(t,y(\varphi_1(t))\right)\right|=\frac{1}{t+1}\left|\left\{\left[\frac{x(t)}{4+x(t)}\right]\right\}-\left\{\left[\frac{y(t)}{4+y(t)}\right]\right\}\right|$

$$\leq\frac{1}{t+1}\left|\left\{\left[\frac{|x(t)-y(t)|+|y(t)|}{4+|x(t)-y(t)|+|y(t)|}\right]\right\}-\left\{\left[\frac{y(t)}{4+|x(t)-y(t)|+|y(t)|}\right]\right\}\right|$$

$$\leq\frac{1}{t+1}\left[\frac{|x(t)-y(t)|}{4+|x(t)-y(t)|+|y(t)|}\right]$$

$$\leq\frac{1}{t+1}\left[\frac{|x(t)-y(t)|}{4+|x(t)-y(t)|}\right]$$

*iv)* For hypothesis$(d)$, taking

$$h(t)=\frac{s^2}{t^6}$$

It is continuous on $\mathcal{R}_+$ .

Implies $g(s,x(\varphi_2(s)))+h(s,x(\varphi_3(s)))\leq h(t)$

That is

$$\frac{t}{t^7(t^2-2+x(8t))}-\frac{t}{t^7(2t+x(t))}\leq\frac{s^2}{t^6}$$

*v)* To show the hypothesis $(e)$is satisfied

$$v(t)=\int_0^t \frac{h(t)}{(t-s)^{1-\zeta}}ds=\int_0^t \frac{\frac{s^2}{t^6}}{(t-s)^{1-\zeta}}ds=\frac{1}{t^6}\int_0^t \frac{s^2}{(t-s)^{1-\zeta}}ds$$

$$=\frac{1}{t^6}\int_0^t s^2(t-s)^{\zeta-1}ds$$

$$=\frac{2\,t^{\zeta+2}}{t^6\zeta(\zeta+1)(\zeta+2)}$$

$$=\frac{2}{t^{4-\zeta}\zeta(\zeta+1)(\zeta+2)}\rightarrow 0\; as\; t\rightarrow\infty$$

$v(t)$is continuous and bounded on $\mathcal{R}_+$ and vanish at infinity.

As all the conditions$(a)$-$(e)$ are satisfied. Hence by theorem (3.1) above problem has a solution.

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