Extremal Solutions of a Non-linear Quadratic Integral Equations

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Abstract: *we prove using hybrid fixed point theorem in banach algebra an existence of solutions to fractional order quadratic integral equation in$\mathcal{R}_+$. Also locally attractivity results and extremal solutions for fractional order quadratic integral equations is proved. Also one example is considered.*

Keywords: Fixed point theorem, Banach algebra, Quadratic integral equation, existence result, locally attractive solution, Extremal Solution.

1. INTRODUCTION:

Fractional Calculus is a generalization of ordinary differentiation and integration to arbitrary order .The subject has its origin in 16^{th} century. During three centuries, the theory of fractional calculus developed as pure theoretical field, useful only for Mathematicians [26] Quadratic functional integral equation has newly received a lot of attention and establishes a meaningful branch of nonlinear analysis. For examples, quadratic integral equations are often applicable in the theory of radiactive transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. Numerous research papers and monographs devoted to quadratic differential and integral equations of fractional order have appeared (see [1-3, 16, 4, 6-7, 12-24]. These papers contain different types of existence results for equations of fractional order. Here we are concerned with the existence of solution for fractional order quadratic functional integral equation also locally attractive solutions. The existence of the maximal and minimal solution of following QIE will be proved.

2. Statement of the Problem:

Let $\alpha, \zeta \in (0,1)$and $\mathcal{R}$denote the real numbers whereas $\mathcal{R}_+$ be the set of nonnegative numbersi.e. $\mathcal{R}_+ = [0, \infty) \subset \mathcal{R}$

Consider the following quadratic integral equations of fractional order

$$x(t) = \left[\frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(t,x(s))}{(t-s)^{1-\alpha}}\, ds\right]\left[q(t) + \frac{1}{\Gamma(\zeta)}\int_0^t \frac{g(s,x(s))}{(t-s)^{1-\zeta}}\, ds\right] \forall t \in \mathcal{R}_+ \qquad (2.1)$$

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Where $q: \mathcal{R}_+ \to \mathcal{R}$, $f(t,x) = f: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$, $g(t,x) = g: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ and By a solution of the (2.1) we mean a function $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{R})$ that satisfies (2.1) on $\mathcal{R}_+$. Where $\mathcal{BC}(\mathcal{R}_+, \mathcal{R})$ is the space of continuous and bounded real-valued functions defined on $\mathcal{R}_+$

In this paper, we prove the locally attractive of the solutions for QIE (2.1) employing a classical hybrid fixed point theorem of B.C.Dhage [4]. In the next section, we collect some preliminary definitions and auxiliary results that will be used in the follows.

3. Preliminaries:

Let $X = \mathcal{BC}(\mathcal{R}_+, \mathcal{R})$ be Banach algebra with norm $\|.\|$ and let Ω be a subset of X. Let a mapping $\mathcal{A} : X \to X$ be an operator and consider the following operator equation in X, namely,

$$x(t) = (\mathcal{A}x)(t)\, t \in \mathcal{R}_+ \tag{3.1}$$

Below we give different characterizations of the solutions for operator equation (3.1) on $\mathcal{R}_+$.

Definition 3.1[17]: The solution $x(t)$ of the equation (3.1) is said to be locally attractive if there exists an closed ball $B_r[0]$ in $\mathcal{BC}(\mathcal{R}_+, \mathcal{R})$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (3.1) belonging to $B_r[0] \cap \Omega$ such that

$$\lim_{t\to\infty}\left(x(t) - y(t)\right) = 0 \tag{3.2}$$

Definition 3.2[17]: Let X be a Banach space. A mapping $\mathcal{A} : X \to X$ is called Lipschitz if there is a constant $\alpha > 0$ such that $\|\mathcal{A}x - \mathcal{A}y\| \le \alpha\|x - y\|$ for all $x, y \in X$ If $\alpha < 1$ then $\mathcal{A}$ is called a contraction on X with the contraction constant α.

Definition 3.3: (Dugundji and Granas [12]). An operator $\mathcal{A}$ on a Banach space X into itself is called Compact if for any bounded subset S of X, $\mathcal{A}$(S) is a relatively compact subset of X. If $\mathcal{A}$ is continuous and compact, then it is called completely continuous on X. Let X be a Banach space with the norm $\|.\|$ and Let $\mathcal{A} : X \to X$ be an operator (in general nonlinear). Then $\mathcal{A}$ is called

(i) Compact if $\mathcal{A}(X)$ is relatively compact subset of X;

(ii) Totally bounded if $\mathcal{A}(S)$ is a totally bounded subset of X for any bounded subset S of X

(iii) Completely continuous if it is continuous and totally bounded operator on X.

It is clear that every compact operator is totally bounded but the converse need not be true.

The solutions of (2.1) in the space $\mathcal{BC}(\mathcal{R}_+, \mathcal{R})$ of continuous and bounded real-valued functions defined on $\mathcal{R}_+$. Define a standard supremum norm $\|.\|$ and a multiplication "." in $\mathcal{BC}(\mathcal{R}_+, \mathcal{R})$ by $\|x\| = \sup\{|x(t)| : t \in \mathcal{R}_+\}$ (3.3)

$$(xy)(t) = x(t)y(t) \quad t \in \mathcal{R}_+ \tag{3.4}$$

Clearly, $\mathcal{BC}(\mathcal{R}_+, \mathcal{R})$ becomes a Banach space with respect to the above norm and the multiplication in it. By $\mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ we denote the space of Lebesgueintegrable functions on $\mathcal{R}_+$ with the norm $\|.\|_{\mathcal{L}^1}$ defined by $\|x\|_{\mathcal{L}^1} = \int_0^\infty |x(t)|dt$ (3.5)

Denote by $\mathcal{L}^1(a, b)$ be the space of Lebesgue-integrable functions on the interval (a, b), which is equipped with the standard norm.

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Let $x \in \mathcal{L}^1(a,b)$ and let $\beta > 0$ be a fixed number.

Definition 3.4[3]: The Riemann-Liouville fractional integral of order β of the function $f(t)$ is defined by the formula $I^\beta f(t) = \frac{1}{\Gamma(\beta)}\int_0^t \frac{f(s)}{(t-s)^{1-\beta}}ds \quad t \in (a,b)$ (3.6)

Where $\Gamma(\beta)$ denote the gamma function.

It may be shown that the fractional integral operator I^β transforms the space $\mathcal{L}^1(a,b)$ into itself and has some other properties (see [18-25])

Definition 3.5: A set $A \subseteq [a,b]$ is said to be measurable if $m^* A = m_* A$. In this case we define mA, the measure of A as $mA = m^* A = m_* A$

If A_1 and A_2 are measurable subsets of $[a,b]$ then their union and their intersection is also measurable.

Clearly every open or closed set in R is measurable.

Definition 3.6: Let f be a function defined on [a,b]. Then f is measurable function if for each $\alpha \in R$, the set $\{x: f(x) > \alpha\}$ is measurable set .

i.e. f is measurable function if for every real number α the inverse image of (α, ∞) is an open set .

As (α, ∞) is an open set and if f is continuous , then inverse image under f of (α, ∞) is open Open sets being measurable , hence every continuous function is measurable.

Definition 3.6: A sequence of functions $\{f_n\}$ is said to converge uniformly on an interval $[a,b]$ to a function f if for any $\epsilon > 0$ and for all $x \in [a,b]$ there exists an integer N (dependent only on ϵ) such that for all $x \in [a,b]$

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N$$

Definition 3.7: The Family F is Equicontinuous at a point $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ a such that $d(f(x_0), f(x)) < \varepsilon$ for all $f \in F$ and all x that $d(x_0, x) < \delta$.

The family is point wise equicontinuous if it is Equicontinuous at each point of X.

The family is uniformly Equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ a such that $d(f(x_1), f(x_2)) < \varepsilon$ for all $f \in F$ and all $x_1, x_2 \in X$ such that $d(x_1, x_2) < \delta$.

Theorem 3.1: (Arzela-Ascoli theorem (7)): If every uniformly bounded and equi-continuous sequence $\{f_n\}$ of functions in $\mathcal{C}(J,\mathcal{R})$, then it has a convergent subsequence.

Theorem 3.2[7]: A metric space X is compact iff every sequence in X has a convergent subsequence.

We employ a hybrid fixed point theorem of Dhage [4] for proving the existence result.

Theorem 3.3 :(Dhage [4]). Let S be a closed-convex and bounded subset of the Banach space X and let $\mathcal{A},\mathcal{B}: S \to X$ be two operators satisfying:

(a) $\mathcal{A}$ is Lipschitz with the Lipschitz constant k,
(b) $\mathcal{B}$ is completely continuous,
(c) $\mathcal{A}x\mathcal{B}x \in S$ for all $,x \in S$ and
(d) $Mk < 1$ Where $M = \|\mathcal{B}(S)\| = \sup\{\|\mathcal{B}x\|: x \in S\}$

Then the operator equation $\mathcal{A}x\mathcal{B}x = x$ has a solution

4. Existence results:

Definition 4.1[7]: A mapping $g: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is said to be Caratheodory if

1. $t \to g(t,x)$ is measurable for all $x \in \mathcal{R}$, and
2. $x \to g(t,x)$ is continuous almost everywhere for $t \in \mathcal{R}_+$

   Again a caratheodory function g is called $\mathcal{L}^1$-Caratheodory if
3. for each real number $r > 0$ there exists a function $h_r \in \mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ such that $|g(t,x)| \leq h_r(t)$ a.e. $t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$ with $|x| \leq r$

Finally, a Caratheodory function $g(t,x)$ is called $\mathcal{L}^1_{\mathcal{R}}$ – Caratheodory if

4. there exist a function $h \in \mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ such that $|g(t,x)| \leq h(t)$

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a.e. $t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$

Throughout this paper, we assume the following Hypothesis

(H_1)The function $f: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is continuous and bounded with bound

$F = \underbrace{sup}_{(t,x)\in\mathcal{R}_+\times\mathcal{R}} |f(t,x)|$there exists a bounded function $l : \mathcal{R}_+ \to \mathcal{R}_+$ with bound L Satisfying

$|f(t,x) - f(t,y)| \le \boldsymbol{l(t)} \frac{\Gamma(\alpha+1)}{(t)^\alpha} | x - y| t \in \mathcal{R}_+$ for all $x, y \in \mathcal{R}$

$(H_2) q: \mathcal{R}_+ \to \mathcal{R}$ is continuous function on $\mathcal{R}_+$; also$\lim_{t\to\infty} q(t) = 0$

(H_3)The functions$f, g: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$satisfy caratheodory condition (i.e.measrable in t for all$x \in \mathcal{R}$and continuous in x for all$t \in \mathcal{R}_+$) and there exist function h_1 , $h_2 \in \mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ Such that$f(t,x) \le h_1\,(t)$ and $g(t,x) \le h_2\,(t) \forall (t,x) \in \mathcal{R}_+ \times \mathcal{R}$

(H_4)The uniform continuous function $v_i : \mathcal{R}_+ \to \mathcal{R}_+$ for i= 1,2defined by the formulas

$$v_1\,(t) = \int_0^t \frac{h_1\,(s)}{(t-s)^{1-\alpha}}\, ds\,, v_2\,(t) = \int_0^t \frac{h_2\,(s)}{(t-s)^{1-\zeta}}\, ds$$

is bounded on $\mathcal{R}_+$ and vanishes at infinity, that is,$\lim_{t\to\infty} v_i\,(t) = 0$

Remark 4.1: Note that if the hypothesis$(H_2)$and (H_3)hold, then there exist constants $K_1 > 0$ and $K_2, K_3 > 0$ such that: $K_1 = sup\{q(t): t \in \mathcal{R}_+\}$,

$$K_2 = \underbrace{sup}_{t\ge0} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h_1\,(s)}{(t-s)^{1-\alpha}}\, ds\,, \qquad K_3 = \underbrace{sup}_{t\ge0} \frac{1}{\Gamma(\zeta)} \int_0^t \frac{h_2\,(s)}{(t-s)^{1-\zeta}}\, ds$$

Theorem 4.1: Let the assumptions [(H_1)-(H_4)] are satisfied. Furthermore if$L(K_1 + K_3) < 1$, where K_1,$K_2$and K_3 are defined remark (4.1), Then the equation (2.1) has a solution in the space $\mathcal{BC}(\mathcal{R}_+, \mathcal{R})$and solutions of the equation (2.1)are locally attractive on$\mathcal{R}_+$ .Moreover (2.1) has maximal and minimal solutions.

Proof: By a solution of the (2.1) we mean a continuous function$x: \mathcal{R}_+ \to \mathcal{R}$that satisfies (2.1) on $\mathcal{R}_+$.

Let $X = \mathcal{BC}(\mathcal{R}_+, \mathcal{R})$be Banach Algebras of all continuous and bounded real valuedfunction on $\mathcal{R}_+$ with the norm$\|x\| = \underbrace{sup}_{t\in\mathcal{R}_+} |x(t)|$ (4.1)

We show that existence of solution for (2.1) under some suitable conditions on the functions involved in (2.1).

Consider the closed ball $B_r[0]$ in X centered at origin 0 and of radius$r$, where

$$r = K_2[K_1 + K_3] > 0$$

Let us define two operators $\mathcal{A}$ and B on $B_r[0]$ by

$$\mathcal{A}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(t, x(s))}{(t-s)^{1-\alpha}}\, ds \tag{4.2}$$

and $\mathcal{B}x(t) = \left[q(t) + \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g(s, x(s))}{(t-s)^{1-\zeta}}\, ds\right] \forall t \in \mathcal{R}_+$ (4.3)

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The function q is continuous on $\mathcal{R}_+$, the function $\mathcal{B}x$ is also continuous and bounded in view of hypotheses $(H_2 - H_3)$The mapping $\mathcal{A}$ is well defined (Since the hypotheses (H_1)) holds)and the function $\mathcal{A}x$ is continuous and bounded on$\mathcal{R}_+$.
Therefore$\mathcal{A}$ and $\mathcal{B}$ define the operators $\mathcal{A},\mathcal{B}: B_r[0] \to X$.we shall show that $\mathcal{A}$ and$\mathcal{B}$ satisfy all the requirements of theorem (3.3) on $B_r[0]$.
Step I: Firstly, it is easy to show that $\mathcal{A}$ is Lipschitz on . Let$x, y \in X$ be arbitrary, and then by hypothesis (H_2), we get

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = \left|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(t,x(s))}{(t-s)^{1-\alpha}}\,ds - \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(t,y(s))}{(t-s)^{1-\alpha}}\,ds\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-s)^{1-\alpha}}\left| f(t,x(s)) - f(t,y(s))\right|ds$$

$$\leq l(t)\frac{\Gamma(\alpha+1)}{(t)^\alpha}\frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-s)^{1-\alpha}}|\ x(t) - y(t)\ |ds$$

$$\leq l(t)\frac{\Gamma(\alpha+1)}{(t)^\alpha}|\ x(t) - y(t)\ |\frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-s)^{1-\alpha}}\,ds$$

$$\leq l(t)\frac{\Gamma(\alpha+1)}{(t)^\alpha}|\ x(t) - y(t)\ |\frac{1}{\Gamma(\alpha)}\left[\frac{(t-s)^\alpha}{\alpha}\right]_0^t$$

$$\leq l(t)\frac{\Gamma(\alpha+1)}{t^\alpha}|\ x(t) - y(t)\ |\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$\leq L|\ x(t) - y(t)\ |\text{for all } t \in \mathcal{R}_+ \tag{4.4}$$

Taking supremum over t

$$\|\mathcal{A}x - \mathcal{A}y\| \leq L\|x - y\|\text{for all} \quad x, y \in B_r[0] \tag{4.5}$$

Gives that $\mathcal{A}$ is Lipschitz on X with the Lipschitz constant L.

Step II: Now we show that$\mathcal{B}$ is completely continuous operator on$B_r[0]$.

Firstly we show that $\mathcal{B}$ is continuous on $B_r[0]$.

Case I: let us assume that, $t \in [0, T]$ then evaluating we obtain the following estimate

$$|(\mathcal{B}x)t - (\mathcal{B}y)t| \leq \left|q(t) + \frac{1}{\Gamma(\zeta)}\int_0^t \frac{g(s,x(s))}{(t-s)^{1-\zeta}}ds - q(t) - \frac{1}{\Gamma(\zeta)}\int_0^t \frac{g(s,y(s))}{(t-s)^{1-\zeta}}ds\right|$$

$$\leq \left|\frac{1}{\Gamma(\zeta)}\int_0^T \frac{g(s,x(s))}{(t-s)^{1-\zeta}}ds - \frac{1}{\Gamma(\zeta)}\int_0^T \frac{g(s,y(s))}{(t-s)^{1-\zeta}}ds\right|$$

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$$\leq \frac{1}{\Gamma(\zeta)}\left[\int_0^T \frac{|g(s,x(s)) - g(s,y(s))|}{(t-s)^{1-\zeta}} ds\right]$$

$$\leq \frac{1}{\Gamma(\zeta)}\left[\int_0^T \frac{w_r^T\,(h_2,\epsilon)}{(t-s)^{1-\zeta}} ds\right]$$

$$\leq \frac{1}{\Gamma(\zeta)}\left[\frac{w_r^T\,(h_2,\epsilon)}{\zeta} T^\zeta ds\right]$$

$$\leq \left[\frac{w_r^T\,(h_2,\epsilon)}{\Gamma(\zeta+1)} T^\zeta ds\right] \tag{4.6}$$

Where $w_r^T(g,\epsilon) = \sup\{|g(s,x) - g(s,y)| : s \in [0,T]; x,y \in [-r,r], |x-y| \leq \epsilon\}$
Therefore, from the uniform continuity of the function $g(t,x)$ on the set $[0,T] \times [-r,r]$ .we derive that $w_r^T(g,\epsilon) \to 0$as $\varepsilon \to 0$.

Case II: Suppose that $t \geq T$ there exist $T > 0$ and let us fix arbitrary$\varepsilon > 0$ and take $x,y \in B_r[0]$ such that$\|x-y\| \leq \varepsilon$Then

$$|(\mathcal{B}x)t - (\mathcal{B}y)t| \leq \left|q(t) + \frac{1}{\Gamma(\zeta)}\int_0^t \frac{g(s,x(s))}{(t-s)^{1-\zeta}}\ ds - q(t) - \frac{1}{\Gamma(\zeta)}\int_0^t \frac{g(s,y(s))}{(t-s)^{1-\zeta}} ds\right|$$

$$\leq \left|\frac{1}{\Gamma(\zeta)}\int_0^t \frac{g(s,x(s))}{(t-s)^{1-\zeta}} ds - \frac{1}{\Gamma(\zeta)}\int_0^t \frac{g(s,y(s))}{(t-s)^{1-\zeta}} ds\right|$$

$$\leq \frac{1}{\Gamma(\zeta)}\left[\int_0^t \frac{|g(s,x(s))|}{(t-s)^{1-\zeta}} ds + \int_0^t \frac{|g(s,y(s))|}{(t-s)^{1-\zeta}} ds\right]$$

$$\leq \frac{1}{\Gamma(\zeta)}\left[\int_0^t \frac{h_2(s)}{(t-s)^{1-\zeta}} ds + \int_0^t \frac{h_2\,(s)}{(t-s)^{1-\zeta}} ds\right]$$

$$\leq \frac{2}{\Gamma(\zeta)}\left[\int_0^t \frac{h_2\,(s)}{(t-s)^{1-\zeta}} ds\right]$$

$$\leq \frac{2\,v_2(t)}{\Gamma(\zeta)} \tag{4.7}$$

Hence we see that There exists T > 0 s.t.

$$v_2(t) \leq \frac{\varepsilon\Gamma(\zeta)}{2} for\ t > T$$

Since ε is an arbitrary,

from (4.7) we derive that

$$|(\mathcal{B}x)t - (\mathcal{B}y)t| \leq \varepsilon \tag{4.8}$$

Now combining the case I and II, we conclude that the operator B is continuous operator on closed ball $B_r[0]$ in to itself.
Step III: Next we show that $\mathcal{B}$is compact on $B_r[0]$.

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(A) First prove that every sequence$\{\mathcal{B}x_n\}$ in $\mathcal{B}(B_r[0])$ has a uniformly bounded sequence in $\mathcal{B}(B_r[0])$. Now by $(H_2)-(H_3)$-(H_4)

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$$|(\mathcal{B}x_n)t| = \left|q(t) + \frac{1}{\Gamma(\zeta)}\int_0^t \frac{g(s, x_n(s))}{(t-s)^{1-\zeta}} ds\right|$$

$$|(\mathcal{B}x_n)t| \le |q(t)| + \frac{1}{\Gamma(\zeta)}\int_0^t \frac{|g(s, x_n(s))|}{(t-s)^{1-\zeta}} ds$$

$$|(\mathcal{B}x_n)t| \le |q(t)| + \frac{1}{\Gamma(\zeta)}\int_0^t \frac{h_2(s)}{(t-s)^{1-\zeta}} ds$$

$$|(\mathcal{B}x_n)t| \le |q(t)| + \frac{v_2(t)}{\Gamma(\zeta)}$$

$$|(\mathcal{B}x_n)t| \le K_1 + K_3 \forall t \in \mathcal{R}_+ \tag{4.9}$$

Taking supremum over t, we obtain $\|\mathcal{B}x_n\| \le K_1 + K_3 \forall n \in N$

This shows that $\{\mathcal{B}x_n\}$ is a uniformly bounded sequence in$\mathcal{B}(B_r[0])$.

(B) Now we proceed to show that sequence$\{\mathcal{B}x_n\}$is also equicontinuous.

Let$\varepsilon > 0$ b be given. Since$\lim_{t\to\infty} q(t) = 0$ there is constant $T > 0$ such that

$|q(t)| < \varepsilon/2$ for all$t \ge T$

Case I: Ift_1 , $t_2 \in [0, T]$then we have

$$|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1|$$

$$\le \left|q(t_2) + \frac{1}{\Gamma(\zeta)}\int_0^{t_2} \frac{g(s, x_n(s))}{(t_2-s)^{1-\zeta}} ds - q(t_1) - \frac{1}{\Gamma(\zeta)}\int_0^{t_1} \frac{g(s, x_n(s))}{(t_1-s)^{1-\zeta}} ds\right|$$

$$\le |q(t_2) - q(t_1)| + \left|\frac{1}{\Gamma(\zeta)}\int_0^{t_2} \frac{|g(s, x_n(s))|}{(t_2-s)^{1-\zeta}} ds - \frac{1}{\Gamma(\zeta)}\int_0^{t_1} \frac{|g(s, x_n(s))|}{(t_1-s)^{1-\zeta}} ds\right|$$

$$\le |q(t_2) - q(t_1)| + \left|\frac{1}{\Gamma(\zeta)}\int_0^{t_2} \frac{h_2(s)}{(t_2-s)^{1-\zeta}} ds - \frac{1}{\Gamma(\zeta)}\int_0^{t_1} \frac{h_2(s)}{(t_1-s)^{1-\zeta}} ds\right|$$

$$\le |q(t_2) - q(t_1)| + \frac{1}{\Gamma(\zeta)}\left|\int_0^{t_2} \frac{h_2(s)}{(t_2-s)^{1-\zeta}} ds - \int_0^{t_1} \frac{h_2(s)}{(t_1-s)^{1-\zeta}} ds\right|$$

$$\le |q(t_2) - q(t_1)| + \frac{1}{\Gamma(\zeta)}|v_2(t_2) - v_2(t_1)| \tag{4.10}$$

from the uniform continuity of the function $q(t), v(t)$ on $[0, T]$, we get $|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1| \to 0$ as$t_1 \to t_2$

Case II: If t_1 , $t_2 \ge T$ then we have

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$|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1|$

$$\leq \left| q(t_2) + \frac{1}{\Gamma(\zeta)} \int_0^{t_2} \frac{g(s, x_n(s))}{(t_2 - s)^{1-\zeta}} ds - q(t_1) - \frac{1}{\Gamma(\zeta)} \int_0^{t_1} \frac{g(s, x_n(s))}{(t_1 - s)^{1-\zeta}} ds \right|$$

$$\leq |q(t_2) - q(t_1)| + \left| \frac{1}{\Gamma(\zeta)} \int_0^{t_2} \frac{|g(s, x_n(s))|}{(t_2 - s)^{1-\zeta}} ds - \frac{1}{\Gamma(\zeta)} \int_0^{t_1} \frac{|g(s, x_n(s))|}{(t_1 - s)^{1-\zeta}} ds \right|$$

$$\leq |q(t_2) - q(t_1)| + \left| \frac{1}{\Gamma(\zeta)} \int_0^{t_2} \frac{|g(s, x_n(s))|}{(t_2 - s)^{1-\zeta}} ds \right| + \left| \frac{1}{\Gamma(\zeta)} \int_0^{t_1} \frac{|g(s, x_n(s))|}{(t_1 - s)^{1-\zeta}} ds \right|$$

$$\leq |q(t_2) - q(t_1)| + \frac{v_2(t_2)}{\Gamma(\zeta)} + \frac{v_2(t_1)}{\Gamma(\zeta)} \leq 0 + \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$\leq \epsilon as t_1 \to t_2. \qquad (4.11)$$

Case III: If $t_1, t_2 \in \mathcal{R}_+$ With $t_1 < T < t_2$ then we have

$|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1| \leq |\mathcal{B}x_n(t_2) - \mathcal{B}x_n(T)| + |\mathcal{B}x_n(T) - \mathcal{B}x_n(t_1)|$ (4.12)

Now if $t_1 \to t_2$ $then t_1 \to$T and T$\to t_2$

Therefore, $|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(T)| \to 0$, $|\mathcal{B}x_n(T) - \mathcal{B}x_n(t_1)| \to 0$

and so $|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1| \to 0$ $as t_1 \to t_2$ for all $t_1, t_2 \in \mathcal{R}_+$(4.13)

Hence $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in $\mathcal{B}(B_r[0])$.

So applying Arzela-Ascoli theorem (see [9]) we say that $\{\mathcal{B}x_n\}$has a uniformly convergent subsequence in $\mathcal{B}(B_r[0])$ and consequently $\mathcal{B}(B_r[0])$ is a relatively compact subset of X. This shows that $\mathcal{B}$ is compact operator on $B_r[0]$. Hence by Dugungi $\mathcal{B}$ is completely continuous on $B_r[0]$.

Step IV: Next we show that $\mathcal{A}x\mathcal{B}x \in B_r[0]$ for all $x \in B_r[0]$ is arbitrary, then

$|\mathcal{A}x(t)\mathcal{B}x(t)| \leq |\mathcal{A}x(t)||\mathcal{B}x(t)|$

$$|\mathcal{A}x(t)\mathcal{B}x(t)| \leq \left| \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(t, x(s))}{(t-s)^{1-\alpha}} ds \right| \left| \left[q(t) + \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g(s, x(s))}{(t-s)^{1-\zeta}} ds \right] \right|$$

$$\leq \left[\frac{1}{\Gamma(\alpha)} \int_0^t \frac{h_1(s)}{(t-s)^{1-\alpha}} ds \right] \left[|q(t)| + \frac{1}{\Gamma(\zeta)} \int_0^t \frac{h_2(s)}{(t-s)^{1-\zeta}} ds \right]$$

$$\leq \frac{v_1(t)}{\Gamma(\alpha)} \left[|q(t)| + \frac{v_2(t)}{\Gamma(\zeta)} \right]$$

$\leq K_2[K_1 + K_3] = r$ for all t in $\mathcal{R}_+$(4.14)

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Taking the supremum over t, we obtain $\| \mathcal{A}x\mathcal{B}x \| \leq r$ for all $x \in B_r[0]$.

Hence $\mathcal{A}x\mathcal{B}x \in B_r[0]$

Hence hypothesis (c) of Theorem (3.3) holds.

Also we have $M = \| \mathcal{B}(B_r[0]) \| = \sup\{ \| \mathcal{B}x \| : x \in B_r[0]\}$

$$= sup\{ \underbrace{sup}_{t\geq 0} \{|q(t)| + \frac{1}{\Gamma(\zeta)} \int_0^t \frac{|g(s,x(s))|}{(t-s)^{1-\zeta}} ds\} : x \in B_r[0] \}$$

$$= sup\{\underbrace{sup}_{t\geq 0} \{|q(t)| + \frac{1}{\Gamma(\zeta)} \int_0^t \frac{h_2(s)}{(t-s)^{1-\zeta}} ds\} : x \in B_r[0]\}$$

$$\leq \underbrace{sup}_{t\geq 0} \{|q(t)| + \underbrace{sup}_{t\geq 0} \frac{v_2(t)}{\Gamma(\zeta)} \leq K_1 + K_3 (4.15)$$

Therefore $ML = L\,(K_1 + K_3) < 1$

Now Appling Theorem 3.3 to shows that (2.1) has a solution on$\mathcal{R}_+$.

LOCAL ATTRACTIVITY OF THE SOLUTIONS

In this section we show the local attractivity of the solutions for (2.1).

Let x and y be any two solutions of the (2.1) in $B_r[0]$ defined on $\mathcal{R}_+$. Then we have,

$$|x(t) - y(t)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(t,x(s))}{(t-s)^{1-\alpha}} ds \left[q(t) + \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g(s,x(s))}{(t-s)^{1-\zeta}} ds \right] - \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(t,y(s))}{(t-s)^{1-\alpha}} ds \left[q(t) + \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g(s,y(s))}{(t-s)^{1-\zeta}} ds \right] \right|$$

$$|x(t) - y(t)| \leq \left| \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(t,x(s))}{(t-s)^{1-\alpha}} ds \left[q(t) + \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g(s,x(s))}{(t-s)^{1-\zeta}} ds \right] \right| + \left| \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(t,y(s))}{(t-s)^{1-\alpha}} ds \left[q(t) + \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g(s,y(s))}{(t-s)^{1-\zeta}} ds \right] \right|$$

$$|x(t) - y(t)| \leq \left| \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(t,x(s))}{(t-s)^{1-\alpha}} ds \right| \left[|q(t)| + \frac{1}{\Gamma(\zeta)} \int_0^t \frac{|g(s,x(s))|}{(t-s)^{1-\zeta}} ds \right] +$$

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$$\left|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(t,y(s))}{(t-s)^{1-\alpha}}\,ds\right|\left[|q(t)| + \frac{1}{\Gamma(\zeta)}\int_0^t \frac{|g(s,y(s))|}{(t-s)^{1-\zeta}}\,ds\right]$$

$$|x(t)-y(t)| \le \frac{1}{\Gamma(\alpha)}\int_0^t \frac{h_1(s)}{(t-s)^{1-\alpha}}\,ds\left[|q(t)| + \frac{1}{\Gamma(\zeta)}\int_0^t \frac{h_2(s)}{(t-s)^{1-\zeta}}\,ds\right] + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{h_1(s)}{(t-s)^{1-\alpha}}\,ds\left[|q(t)| + \frac{1}{\Gamma(\zeta)}\int_0^t \frac{h_2(s)}{(t-s)^{1-\zeta}}\,ds\right]$$

$$|x(t)-y(t)| \le 2\frac{1}{\Gamma(\alpha)}\int_0^t \frac{h_1(s)}{(t-s)^{1-\alpha}}\,ds\left[|q(t)| + \frac{1}{\Gamma(\zeta)}\int_0^t \frac{h_2(s)}{(t-s)^{1-\zeta}}\,ds\right]$$

$$|x(t)-y(t)| \le 2\frac{v_1(t)}{\Gamma(\alpha)}\left[|q(t)| + \frac{v_2(t)}{\Gamma(\zeta)}\right] \tag{4.16}$$

For all$t \in \mathcal{R}_+$.Since $\lim_{t\to\infty} q(t) = 0$ and$\lim_{t\to\infty} v_i(t) = 0$ this gives that $\lim_{t\to\infty} \sup|x(t)-y(t)| = 0$. Thus the (2 .1) has a solution and the solutions are locally attractive on $\mathcal{R}_+$

5. Existence of Extremal Solution:

In this section we show that given equation (2.1) has maximal and minimal solutions:

Definition 5.1: **(Caratheodory case)** A function $\tau:\mathcal{R}\to\mathcal{R}$ is nondecreasing if $\tau(x) \le \tau(y)\forall x,y \in \mathcal{R}$ for which $x \le y$.similarly $\tau(x)$ is increasing in x if $\tau(x) < \tau(y)\forall x,y \in \mathcal{R}$ for which $x < y$.

Definition 5.2: (Chandrabhan) A function $f:\mathcal{R}_+\times\mathcal{R}\to\mathcal{R}$is called *chandrabhan* if

i) The function $(x,y)\to f(x,y,z)$ is measurable for each z $\in \mathcal{R}$

ii) The function $z \to f(x,y,z)$ is non-decreasing for almost each $(x,y) \in \mathcal{R}_+$

Definition 5.3[9, 28]: A closed and non-empty set $\mathcal{K}$ in a Banach Algebra X is called a cone if

i. $\mathcal{K}+\mathcal{K} \subseteq \mathcal{K}$
ii. $\lambda\mathcal{K} \subseteq \mathcal{K}$ for $\lambda \in \mathcal{K}, \lambda \ge 0$
iii. $\{-\mathcal{K}\} \cap \mathcal{K} = 0$ where 0 is the zero element of X.

and is called positive cone if

iv. $\mathcal{K} \circ \mathcal{K} \subseteq \mathcal{K}$

And the notation $\circ$ is a multiplication composition in X.
The Details of Cones and their properties may be found in the monographs like Guo and Lakshmikantham [9] and Heikkila and Lakshmikantham [28]

Definition 5.4: A solution x_M of the integral equation is said to be maximal if for any other solution x to the problem $x(t) \le x_M(t)$ $\quad t \in \mathcal{R}_+$

Again a solution x_m of the integral equation is said to be minimal if for any other solution x to the problem $x_m(t) \le x(t)$ $\quad t \in \mathcal{R}_+$

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Definition 5.5[5]: A mapping $R:[p_1,p_2] \to X$ is said to be nondecreasing or monotone increasing if $x \le y$ implies $Rx \le Ry$ for all $x,y \in [p_1,p_2]$.

Theorem 5.1: Let $\mathcal{K}$ be a cone in a banach algebra X and let $[\overline{x}\,,\underline{x}] \in X$
Suppose $\mathcal{A},\mathcal{B} : [\,\overline{x}\,,\underline{x}\,] \to \mathcal{K}$be two operators such that
a) $\mathcal{A}$is lipschitz with Lipschitz constant α
b) $\mathcal{B}$is totally bounded
c) $x_1\mathcal{B}x_2 \in [\,\overline{x}\,,\underline{x}\,] \qquad \forall x_1\,,x_2 \in [\,\overline{x}\,,\underline{x}\,]$
d) $\mathcal{A}$and$\mathcal{B}$ are nondecreasing
further if the cone $\mathcal{K}$ is positive and normal then the operator equation$\mathcal{A}x\mathcal{B}x = x$ has a least and a greatest positive solution in $[\,\overline{x}\,,\underline{x}\,]$whenever α M < 1
Where M = $\|\mathcal{B}[\,\overline{x}\,,\underline{x}\,]\|\sup\{\,\|\mathcal{B}x : x \in [\,\overline{x}\,,\underline{x}\,]\|\}$
we equip the space $BC(\mathcal{R}_+,\mathcal{R})$ with the order relation $\le$ with the help of the cone defined by
$\mathcal{K} = \{x \in C(\mathcal{R}_+,\mathcal{R}): x(t) \ge 0\ \forall t \in \mathcal{R}_+\}$
Thus $x \le \overline{x}$ iff $x\,(t) \le \overline{x}(\,t) \quad \forall\, x \in \mathcal{R}_+$
It is well known that the cone $\mathcal{K}$ is positive and normal in $BC(\mathcal{R}_+,\mathcal{R})$

We consider another hypothesis
$(\boldsymbol{H_6})$ Suppose $f(t,x) = f:\mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}\,, g(t,x) = g:\mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ are Chandrabhan
$(\boldsymbol{H_7})$ There exists a function $\overline{h} \in L^1(\mathcal{R}_+\,,\mathcal{R})$ such that
$|g(t,x)| \le \overline{h}(t,x) \forall t \in \mathcal{R}_+$ and$x \in \mathcal{R}$
$(\boldsymbol{H_8})$ The given problem has a lower solution $\underline{x}$ and upper solution $\overline{x}$ with $\underline{x} \le \overline{x}$
holds if $L(\boldsymbol{K_1} + \boldsymbol{K_3}) <1$
Theo 5.2: Suppose that the Hypotheses$(\boldsymbol{H_6})$ - $(\boldsymbol{H_8})$ are holds .Then problem (2.1) have a minimal and maximal positive solution on $\mathcal{R}_+$

Proof: Let $X = \mathcal{BC}(\mathcal{R}_+,\mathcal{R})$ and define an order relation$\le$ by the cone $\mathcal{K}$ given by (5.3) clearly $\mathcal{K}$ is normal cone in X. Define the two operators $\mathcal{A}$ and $\mathcal{B}$ on X by (4.2) and (4.3) resp. then (2.1) is equivalent to operator equation $\mathcal{A}x\mathcal{B}x=x$.Now it is shown as in the proof that$\mathcal{A}$ is Lipchitz with Lipchitz constant L and $\mathcal{B}$ is completely continuous operator. Let $x_1{,}x_2 \in [\,\overline{x}\,,\underline{x}\,]$ s.t. $x_1 \le x_2$Then by hypothesis $(\boldsymbol{H_6})$

$$\mathcal{A}x_1\,(\mathrm{t}) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f\big(t,x_1(s)\big)}{(t-s)^{1-\alpha}}\,ds \le \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f\big(t,x_2(s)\big)}{(t-s)^{1-\alpha}}\,ds = \mathcal{A}x_2(t)$$

for all t $\in \mathcal{R}_+$
and

$$\mathcal{B}x_1\,(\mathrm{t}) = \left[q(t) + \frac{1}{\Gamma(\zeta)}\int_0^t \frac{g(s,x_1(s))}{(t-s)^{1-\zeta}}\,ds\right] \le \left[q(t) + \frac{1}{\Gamma(\zeta)}\int_0^t \frac{g(s,x_2(s))}{(t-s)^{1-\zeta}}\,ds\right] = \mathcal{B}x_2\,(\mathrm{t})$$

So $\mathcal{A}$ and $\mathcal{B}$ are non decreasing operator on $[\overline{x}\,,\underline{x}]$
Now

$$\underline{x}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f\big(t,\underline{x}(s)\big)}{(t-s)^{1-\alpha}}\,ds\left[q(t) + \frac{1}{\Gamma(\zeta)}\int_0^t \frac{g(s,\underline{x}(s))}{(t-s)^{1-\zeta}}\,ds\right]$$

$$\underline{x}(t) \le \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f\big(t,x(s)\big)}{(t-s)^{1-\alpha}}\,ds\left[q(t) + \frac{1}{\Gamma(\zeta)}\int_0^t \frac{g(s,x(s))}{(t-s)^{1-\zeta}}\,ds\right]$$

$$\underline{x}(t) \le \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f\big(t,\overline{x}\,(s)\big)}{(t-s)^{1-\alpha}}\,ds\left[q(t) + \frac{1}{\Gamma(\zeta)}\int_0^t \frac{g(s,\overline{x}\,(s))}{(t-s)^{1-\zeta}}\,ds\right]$$

$\underline{x}(t) \le \overline{x}\,(t)$for all $x \in [\,\overline{x}\,,\underline{x}\,]$ and $t \in \mathcal{R}_+$

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Hence $\mathcal{A}x\mathcal{B}x \in [\,\overline{x}\,, \underline{x}\,]$for all $x \in [\,\overline{x}\,, \underline{x}\,]$

$$M = \|B[\overline{x}\,, \underline{x}]\|$$

$$\leq sup|q(t)| + \left|\frac{1}{\Gamma(\zeta)}\int_0^t \frac{g(s,x(s))}{(t-s)^{1-\zeta}}\,ds\,\right|$$

$$\leq sup|q(t)| + \frac{1}{\Gamma(\zeta)}\left|\int_0^t \frac{|g(s,x(s))|}{(t-s)^{1-\zeta}}\,ds\,\right|$$

$$\leq sup|q(t)| + \frac{1}{\Gamma(\zeta)}\int_0^t \frac{h_2\,(s\,)}{(t-s)^{1-\zeta}}\,ds$$

$$LM \;\leq L(\boldsymbol{K_1} + \boldsymbol{K_3}) < 1$$

Thus operator equation has maximal and minimal solution in $[\overline{x}\;, \underline{x}]$.

Thus given problem have maximal and minimal positive solution on $\mathcal{R}_+$.

6. Consider the following quadratic functional Integral equation of type (2.1)

$$x(t) = \left[\frac{1}{\Gamma(\alpha)}\int_0^t \frac{e^{-t}\,x(t)}{(t-s)^{1-\alpha}}\;ds\right]\left[\frac{1}{t^6} + \frac{1}{\Gamma(\zeta)}\int_0^t \frac{(t-s)^{\zeta-1}}{t^4[1+x(t)]}\,ds\right]\forall t \in \;\; \mathcal{R}_+$$

Here$\alpha = \frac{1}{2}\zeta = \frac{1}{4}$ $\;f(t,x(s)) = e^{-t}\,x(t)$ $\;g(s,x(s)) = \frac{1}{t^4[1+x(t)]}q(t) = \frac{1}{t^6}$

$(\mathcal{H}_1)$Now$|f(t,x(t)) - f(t,y(t))|$

$$= |\{e^{-t}\,x(t)\,\} - \{e^{-t}\,y(t)\}|$$

$$= |e^{-t}[x(t) - y(t)]|$$

$$\leq |e^{-t}||x(t) - y(t)|$$

$$\leq k(t)|x(t) - y(t)|$$

$$\leq F|x(t) - y(t)|$$

Since $k(t) = e^{-t}$ say which has bound F on $\mathcal{R}_+$.

$(\mathcal{H}_2)$ $\;q(t) = \dfrac{1}{t^6}$ is continuous on R_+ and $\lim_{t\to\infty} q(t) = 0$

$(\mathcal{H}_3)$ Take $h_1(t) = \frac{s}{t^2}$, it is continuous on $\mathcal{R}_+$.

Implies $f(t,x(t)) \leq h_1(t)$

That is $e^{-t}x(t) \leq \frac{s}{t}$

Take $h_2(t) = \frac{s}{t^3}$, it is continuous on $\mathcal{R}_+$.

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Implies $g\big(t, x(t)\big) \leq h_2(t)$

That is $\frac{1}{t^4[1+x(t)]} \leq \frac{s}{t^3}$

$$(\mathcal{H}_4)\ v_1(t) = \int_0^t \frac{h_1(s)}{(t-s)^{1-\alpha}}\, ds = \int_0^t \frac{\frac{s}{t^2}}{(t-s)^{1-\alpha}}\, ds = \frac{1}{t^2}\int_0^t \frac{s}{(t-s)^{1-\alpha}}\, ds$$

$$= \frac{1}{t^2}\int_0^t s.(t-s)^{\alpha-1}\, ds = \frac{1}{t^2}\left[s.\frac{(t-s)^{\alpha}}{-\alpha}\right]_0^t + \frac{1}{t^2}\left[\frac{(t-s)^{\alpha+1}}{-(\alpha+1)}\right]_0^t = \frac{1}{t^2}[0 - \frac{t^{\alpha+1}}{-(\alpha+1)}]$$

$$= \frac{1}{t^{1-\alpha}(\alpha+1)} \ \to 0\ as\ t \to \infty$$

$$Again\ \ v_2(t) = \int_0^t \frac{h_2(s)}{(t-s)^{1-\zeta}}\, ds = \int_0^t \frac{\frac{s}{t^3}}{(t-s)^{1-\zeta}}\, ds = \frac{1}{t^3}\int_0^t \frac{s}{(t-s)^{1-\zeta}}\, ds$$

$$= \frac{1}{t^3}\int_0^t s.(t-s)^{\zeta-1}\, ds = \frac{1}{t^3}\left[s.\frac{(t-s)^{\zeta}}{-\zeta}\right]_0^t + \frac{1}{t^3}\left[\frac{(t-s)^{\zeta+1}}{-(\zeta+1)}\right]_0^t = \frac{1}{t^3}[0 - \frac{t^{\zeta+1}}{-(\zeta+1)}]$$

$$= \frac{1}{t^3}\left[\frac{t^{\zeta+1}}{(\zeta+1)}\right] = \frac{1}{t^{2-\zeta}\,(\zeta+1)} \ \to 0\ as\ t \to \infty$$

$v_i(t)$is continuous and bounded on $\mathcal{R}_+$ and vanish at infinity.
It follows that all the conditions $(\mathcal{H}_1)$-$(\mathcal{H}_5)$ are satisfied. Thus by theorem (4.1), above problem (6.1) has a solution $\mathcal{R}_+$.

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