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Existence and Extremal Solution for Fractionalorder Nonlinear Integro-Differential Equation in Banach Algebras

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Abstract: In this Paperwe study theexistence the solution for a fractional order nonlinear integro-differential equation in Banach Algebras.Wemake use of the standard tools of the hybrid fixed Point theory for three operators to establish the main result. We also proved theexistence solutionsare locally attractive in $\mathcal{R}_+$ and extremal result are also proved. Finally, our results are illustrated by a concrete example.

Keywords: Banach Algebras, Integro-Differential Equation, Existence Result, Fixed Point Theorem,Locally Attractive Solution, Extremal Solution.

AMS Subject Classification: 26A23, 34K10, 34G20, 37K37, 47H10.

1. Introduction:

Fractional Calculus is a generalization of ordinary differential and integration to arbitrary (non-integer) order. The subject has its origin in the 1600s.During three centuries, the theory of fractional calculus developed as a pure theoretical field, useful only for mathematicians. We can see that the tremendous development of theory of fractional calculus occurred in last 3-4 decades.Fractional differentiation proved very useful in various fields of applied sciences and engineering. The "bible" of fractional calculus is the book of Samko, Kilbas and Marichev [34]. Several definitions of fractional derivatives and integral are available in the literature, including the Riemann-Liouville, Caputo derivatives and integral [1, 2, 3, 9, 11-13]. The combination of fractional calculus and integral equations may introduce more effective tool for analysis.

Fractional differential and integral equations have also been studied by several researchers. This class of equation involves the fractional derivative and integral of an unknown function. Some recent results on fractional order differential and integral equations can be found in a series of papers (see[7,14-17]).

In this paper, we present the existence results along with the locally attractivity and extremal solutions forfractional order nonlinearfunctional integro-differential equation in Banach Algebras. Finally, we present an example illustrating the applicability of the imposed conditions.

2. Statement of the Problem:

Let $\zeta, \delta \in (0,1)$, $\mathcal{R}$ denote the real numbers whereas $\mathcal{R}_+$ be the set of nonnegative numbersi.e. $\mathcal{R}_+ = [0,\infty) \subset \mathcal{R}$

Consider thefractional order nonlinear functional integro-differential equation

$$D^{\zeta}\left[\frac{D^{\delta}x(t) - \sum_{k=1}^{n} I^{\beta_k} h_k(t, x(\gamma(t)))}{f(t, x(\alpha(t)))}\right] = g\left(t, x(\mu(t)), I^{\rho}x(\tau(t))\right) \quad \forall \quad t \in \mathcal{R}_+ \quad (2.1)$$

$x(0)= 0$ and $D^{\delta}x(0)= 0$

where $0 < \zeta, \delta < 1$, $0 < \zeta + \delta < 2$, $f(t,x) = f: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}\setminus\{0\}$, $g(t,x,y) = g: \mathcal{R}_+ \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ and $h_k: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ with $h_k(0,0) = 0, k = 1,2,.......n$, $\alpha, \gamma, \tau: \mathcal{R}_+ \to \mathcal{R}_+$.

D^{ζ}denotes R-L fractional derivative of order ζ and I^{ρ}denotes R-L fractional integral of order ρ.

By a solution of the (2.1) we mean a function $x \in BC(\mathcal{R}_+, \mathcal{R})$ that satisfies (2.1) on $\mathcal{R}_+$. Where $BC(\mathcal{R}_+, \mathcal{R})$is the space of continuous and bounded real-valued functions defined on $\mathcal{R}_+$.

Applying a hybrid fixed point theorem [5], the existence results forFIDE (2.1) will be obtained.

In section 3 we recall some useful preliminaries. In section 4 we study the existence the solution of the initial value problem (2.1), while in section 5 we deal with the existence of extremal solution of the initial value problem (2.1).Example illustrating the obtained results are presented in section 6.

3. Preliminaries:

Let $X = BC(\mathcal{R}_+, \mathcal{R})$ be Banach algebra with norm $\|.\|$ and let Ω be a subset of X. Let a mapping $\mathcal{A} : X \to X$ be an operator and consider the following operator equation in X, namely,

$$x(t) = (\mathcal{A}x)(t), \quad t \in \mathcal{R}_+ \tag{3.1}$$

Below we give different characterizations of the solutions for operator equation (3.1) on $\mathcal{R}_+$.

We list some precise definitions in the sequel.

Definition 3.1[22]: The solution $x(t)$ of the equation (3.1) is said to be locally attractive if there exists an closed ball $B_r[0]$ in $BC(\mathcal{R}_+, \mathcal{R})$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (3.1) belonging to $B_r[0] \cap \Omega$ suchthat

$$\lim_{t\to\infty}\big(x(t) - y(t)\big) = 0 \tag{3.2}$$

Definition 3.2[22]: Let Xbe a Banach space. A mapping $\mathcal{A} : X \to X$ is called Lipschitz if there is a constant $\alpha > 0$ such that$\|\mathcal{A}x - \mathcal{A}y\| \le \alpha\|x - y\|$ for all $x, y \in X$.

If $\alpha < 1$ then $\mathcal{A}$ is called a contraction on X with the contraction constant α.

Definition 3.3: (Dugundji and Granas [18]). An operator $\mathcal{A}$ on a Banach space X into itself is called Compact if for any bounded subset S of X, $\mathcal{A}$(S) is a relatively compact subset of X. If $\mathcal{A}$ is continuous and compact, then it is called completely continuous on X. Let Xbe a Banach space with the norm $\|.\|$and Let $\mathcal{A} : X \to X$ be an operator (in general nonlinear). Then $\mathcal{A}$ is called

(i) Compact if $\mathcal{A}(X)$ is relatively compact subset of X;

(ii) Totally bounded if $\mathcal{A}(S)$is a totally bounded subset of X for any bounded subset S of X

(iii) Completely continuous if it is continuous and totally bounded operator on X.

It is clear that every compact operator is totally bounded but the converse need not be true.

The solutions of (2.1) in the space $BC(\mathcal{R}_+,\mathcal{R})$ of continuous and bounded real-valued functions defined on $\mathcal{R}_+$. Define a standard supremum norm $\|.\|$ and a multiplication "." in $BC(\mathcal{R}_+,\mathcal{R})$ by $\|x\| = \sup\{|x(t)| : t \in \mathcal{R}_+\}$(3.3)

$(xy)(t) = x(t)y(t) \quad t \in \mathcal{R}_+$ (3.4)

Clearly, $BC(\mathcal{R}_+,\mathcal{R})$ becomes a Banach space with respect to the above norm and the multiplication in it. By $\mathcal{L}^1(\mathcal{R}_+,\mathcal{R})$ we denote the space of Lebesgueintegrable functions on $\mathcal{R}_+$ with the norm $\|.\|_{\mathcal{L}^1}$ defined by

$$\|x\|_{\mathcal{L}^1} = \int_0^\infty |x(t)|dt \tag{3.5}$$

Denote by $\mathcal{L}^1(a,b)$ be the space of Lebesgueintegrable functions on the interval (a, b), which is equipped with the standard norm. Let $x \in \mathcal{L}^1(a,b)$ and let $\beta > 0$ be a fixed number.

Definition 3.4[21]: The Riemann-Liouville fractional integral of order β of the function $x(t)$ is defined by the formula:

$$I^\beta x(t) = \frac{1}{\Gamma(\beta)}\int_0^t \frac{x(s)}{(t-s)^{1-\beta}}ds \qquad t \in (a,b) \tag{3.6}$$

Where $\Gamma(\beta)$ denote the gamma function.

It may be shown that the fractional integral operator I^β transforms the space $\mathcal{L}^1(a,b)$ into itself and has some other properties (see [12-19])

Definition 3.5: Aset A⊆ [a,b] is said to be measurable if $m^* A = m_* A$. In this case we define mA , the measure of A as mA = $m^* A = m_* A$

If $A_1$and A_2 are measurable subsets of [a ,b] then their union and their intersection is also measurable.

Clearly every open or closed set in R is measurable.

Definition 3.6: Let$f$ be a function defined on [a,b].Then f is measurable function if for each $\alpha \in R$,$theset\{x{:}f(x) > \alpha\}$ is measurable set .

i.e. f is measurable function if for every real number α the inverse image of (α,∞) is an open set

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As (α, ∞) is an open set and if f is continuous, then inverse image under $f$of (α, ∞) openOpen sets being measurable , hence every continuous function is measurable.

Definition 3.6:A sequence of functions $\{ f_n \}$is said to converge uniformly on an interval $[a, b]$ to a function f if for any $\epsilon > 0$ and for all $x \in [a, b]$ there exists an integer N (dependent only on ϵ) such that for all $x \in [a, b]$

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N$$

Definition 3.7: The Family F is equicontinuous at a point $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(f(x_0), f(x)) < \varepsilon$ for all $f \in F$ and all x that $d(x_0, x) < \delta$.

The family is point wiseequicontinuous if it is equicontinuous at each pointof X.

The family is uniformly equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ a such that $d(f(x_1), f(x_2)) < \varepsilon$ for all $f \in F$ and all $x_1, x_2 \in X$ such that$d(x_1, x_2) < \delta$.

Lemma 3.1 [17]: Let $q > 0$ and $x \in \mathcal{C}(0, T) \cap \mathcal{L}(0, T)$ Then we have

$$I^q \frac{d^q}{dt^q} x(t) = x(t) - \sum_{j=1}^{n} \frac{(I^{n-q}x)^{(n-j)}(0)}{\Gamma(q - j + 1)} t^{q-j}$$

,where $n - 1 < q < n$.

Theorem 3.1: (Arzela-Ascoli theorem [6]): If every uniformly bounded and equi-continuous sequence $\{f_n\}$ of functions in $\mathcal{C}(\mathcal{R}_+, \mathcal{R})$, then it has a convergent subsequence.

Theorem 3.2[6]: A metric space X is compact iff every sequence in X has a convergent subsequence.

We employ a hybrid fixed point theorem of Dhage [5] for proving the existence result.

Theorem 3.3 :(Dhage [5,16]): Let S be a non empty, convex, closed and bounded subset of the Banach space X and let $\mathcal{A}, \mathcal{C}: X \to X$ and $\mathcal{B}: S \to X$ are two operators satisfying:

a) $\mathcal{A}$ and $\mathcal{C}$ are Lipschitzian with lipschitz constants ζ, η respectively.
b) $\mathcal{B}$ is completely continuous, and
c) $x = \mathcal{A}x\mathcal{B}y + \mathcal{C}x \in S$ for all $y \in S$
d) $\xi M + \eta < 1$ where $M = \|\mathcal{B}(s)\| = sup\{\|\mathcal{B}x\| : x \in S\}$

Then the operator equation $x = \mathcal{A}x\mathcal{B}y + \mathcal{C}x$ has a solution in S.

4. Existence results:

Definition 4.1[6]: A mapping $g:\mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is said to be Caratheodory if

1. $t \to g(t,x)$ is measurable for all $x \in \mathcal{R}$, and
2. $x \to g(t,x)$ is continuous almost everywhere for $t \in \mathcal{R}_+$

Again a caratheodory function g is called $\mathcal{L}^1$-Caratheodory if

3. for each real number $r > 0$ there exists a function $h_r \in \mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ such that $|g(t,x)| \leq h_r(t)$ a.e. $t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$ with $|x| \leq r$

Finally, a Caratheodory function $g(t,x)$ is called $\mathcal{L}^1_{\mathcal{R}}$ – Caratheodory if

4. there exist a function $h \in \mathcal{L}^1(\mathcal{R}_+, \mathcal{R})$ such that $|g(t,x)| \leq h(t)$ a.e. $t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$

For convenience, the function h is referred to as a bound function of g.

Definition 4.1.1[7] :A mapping $g:\mathcal{R}_+ \times \mathcal{R}_+ \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is Caratheodory if:

i) $(t,s) \to g(t,s,x,y)$ is measurable for each $x, y \in \mathcal{R}$ and
ii) $(x,y) \to g(t,s,x,y)$ is continuous almost everywhere for $t \in \mathcal{R}_+$.

Furthermore a Caratheodary function g is $\mathcal{L}^1$ –Caratheodary if:

iii) For each real number $r > 0$ there exists a function $h_r \in \mathcal{L}^1(\mathcal{R}_+ \times \mathcal{R}_+, \mathcal{R})$ such that $|g(t,s,x,y)| \leq h_r(t,s)\, a.e.$ $t \in \mathcal{R}_+$ for all $x, y \in \mathcal{R}$ with $|x|_r \leq r$ and $|y|_r \leq r$.

Finally a caratheodary function g is $\mathcal{L}^1_X$ –caratheodary if:

iv) There exists a function $h \in \forall \mathcal{L}^1(\mathcal{R}_+ \times \mathcal{R}_+, \mathcal{R})$ such that $|g(t,s,x,y)| \leq h(t,s)$, $a.e.$ $t \in \mathcal{R}_+$ for all $x, y \in \mathcal{R}$

For convenience, the function h is referred to as a bound function for g.

Lemma 4.1: Suppose that $\zeta, \delta \in (0,1)$ and the function $f, g, h_k, k = 1,2,3, \ldots n$ satisfying FIDE (2.1). Then x is the solution of the FIDE (2.1) if and only if it is the solution of integral equation

$x(t)$

$$= I^{\delta} f(t, x(\alpha(t))) \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, x(\mu(t)), I^{\rho} x(\tau(t))\right)}{(t-s)^{1-\zeta}} ds + \sum_{k=1}^{n} I^{\beta_k+\delta} h_k(t, x(\gamma(t))) \quad (4.1)$$

for all $t \in \mathcal{R}_+$ and $\zeta, \delta \in (0,1)$.

Proof: Applying the Riemann-Liouville fractional integral of order ζ to both sides of (2.1) we have

$$I^{\zeta} \frac{d^{\zeta}}{dt^{\zeta}} \left\{ \frac{D^{\delta} x(t) - \sum_{k=1}^{n} I^{\beta_k} h_k(t, x(\gamma(t)))}{f(t, x(\alpha(t)))} \right\}_0^t = I^{\zeta} g\left(t, x(\mu(t)), I^{\rho} x(\tau(t))\right)$$

$$\therefore \left\{ \frac{D^{\delta} x(t) - \sum_{k=1}^{n} I^{\beta_k} h_k(t, x(\gamma(t)))}{f(t, x(\alpha(t)))} \right\}_0^t = I^{\zeta} g\left(t, x(\mu(t)), I^{\rho} x(\tau(t))\right)$$

$$\therefore \frac{D^{\delta} x(t) - \sum_{k=1}^{n} I^{\beta_k} h_k(t, x(\gamma(t)))}{f(t, x(\alpha(t)))} = \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, x(\mu(t)), I^{\rho} x(\tau(t))\right)}{(t-s)^{1-\zeta}} ds$$

$$\therefore D^{\delta} x(t) - \sum_{k=1}^{n} I^{\beta_k} h_k(t, x(\gamma(t))) = f(t, x(\alpha(t))) \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, x(\mu(t)), I^{\rho} x(\tau(t))\right)}{(t-s)^{1-\zeta}} ds$$

Since $x(0) = 0$, $h_k(0,0) = 0, f(0,0) \neq 0$

It follows that

$$D^{\delta} x(t) = f(t, x(\alpha(t))) \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, x(\mu(t)), I^{\rho} x(\tau(t))\right)}{(t-s)^{1-\zeta}} ds + \sum_{k=1}^{n} I^{\beta_k} h_k(t, x(\gamma(t))),$$

$t \in \mathcal{R}_+$

$$I^{\delta}D^{\delta}x(t) = I^{\delta}f(t,x(\alpha(t))\frac{1}{\Gamma(\zeta)}\int_0^t\frac{g\left(t,x(\mu(t)),I^{\rho}x(\tau(t))\right)}{(t-s)^{1-\zeta}}ds + I^{\delta}\sum_{k=1}^{n}I^{\beta_k}h_k(t,x(\gamma(t))$$

$$x(t) = I^{\delta}f(t,x(\alpha(t))\frac{1}{\Gamma(\zeta)}\int_0^t\frac{g\left(t,x(\mu(t)),I^{\rho}x(\tau(t))\right)}{(t-s)^{1-\zeta}}ds + \sum_{k=1}^{n}I^{\beta_k+\delta}h_k(t,x(\gamma(t))$$

Applying the semi group Property for $I^{\delta}I^{\beta_k}h_k = I^{\beta_k+\delta}h_k k = 1,2,3,\ldots n$

Conversely differentiate (4.1) of order δ and then ζ with respect to t , we get,

$$\frac{d^{\zeta}}{dt^{\zeta}}\left\{\frac{D^{\delta}x(t)-\sum_{k=1}^{n}I^{\beta_k}h_k(t,x(\gamma(t))}{f(t,x(\alpha(t))}\right\} = \frac{d^{\zeta}}{dt^{\zeta}}\frac{1}{\Gamma(\zeta)}\int_0^t\frac{g\left(t,x(\mu(t)),I^{\rho}x(\tau(t))\right)}{(t-s)^{1-\zeta}}ds$$

$$\Rightarrow\frac{d^{\zeta}}{dt^{\zeta}}\left\{\frac{D^{\delta}x(t)-\sum_{k=1}^{n}I^{\beta_k}h_k(t,x(\gamma(t))}{f(t,x(\alpha(t))}\right\} = \frac{d^{\zeta}}{dt^{\zeta}}I^{\zeta}g\left(t,x(\mu(t)),I^{\rho}x(\tau(t))\right)$$

$$\Rightarrow\frac{d^{\zeta}}{dt^{\zeta}}\left\{\frac{D^{\delta}x(t)-\sum_{k=1}^{n}I^{\beta_k}h_k(t,x(\gamma(t))}{f(t,x(\alpha(t))}\right\} = g\left(t,x(\mu(t)),I^{\rho}x(\tau(t))\right) \quad t\in\mathcal{R}_+$$

We consider the fractional order nonlinear quadratic functional integro-differentialequation (2.1) assuming that the following hypothesis is satisfied.

(H_1) The function $f(t,x) : \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}\setminus\{0\}$ is continuous and bounded with bound

$F = \underbrace{sup}_{(t,x)\in\mathcal{R}_+\times\mathcal{R}}|f(t,x)|$ there exists a bounded function $l : \mathcal{R}_+ \to \mathcal{R}_+$ with bound L Satisfying

$|f(t,x) - f(t,y)| \le l(t)|x-y|, t\in\mathcal{R}_+$ for all $x,y\in\mathcal{R}$

(H_2) The functions $g:\mathcal{R}_+\times\mathcal{R}\times\mathcal{R}\to\mathcal{R}$ satisfy caratheodory condition (i.e.measrable in t for all $x\in\mathcal{R}$ and continuous in x for all $t\in\mathcal{R}_+$) and there exist function $m_1 \in \mathcal{L}^1(\mathcal{R}_+,\mathcal{R})$ such that $g(t,x,y)\le m_1(t)$ $\forall(t,x,y)\in\mathcal{R}_+\times\mathcal{R}\times\mathcal{R}$

Additionally we will assume following conditions are satisfied.

(H_3) The uniform continuous function $v: \mathcal{R}_+ \to \mathcal{R}_+$ defined by the formulas

$v(t) = \int_0^t \frac{m_1(s)}{(t-s)^{1-\zeta}} ds$ is bounded on $\mathcal{R}_+$ and vanishat infinity, that is, $\lim_{t\to\infty} v(t) = 0$

(H_4) The function $h_k: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}, k = 1,2,3 \ldots\ldots n$, with $h_k(0,0,) = 0$ $k = 1,2,3 \ldots\ldots n$ are continuous and there exist positive functions $\lambda_k, k = 1,2,3 \ldots\ldots n$ with bound $\|\lambda_k\|$ such that

$$\left| h_k\Big(t, x\big(\gamma(t)\big)\Big) - h_k\Big(t, y\big(\gamma(t)\big)\Big) \right| \le \lambda_k(t)|x(t) - y(t)|, \forall\, t \in \mathcal{R}_+, x, y \in \mathcal{R}$$

$$\left| h_k\Big(t, x\big(\gamma(t)\big)\Big) \right| \le H_k(t) \quad and \quad \lim_{t\to\infty} H_k(t) = 0$$

(H_5) The functions $\alpha, \gamma, \tau: \mathcal{R}_+ \to \mathcal{R}$ are continuous.

Remark 4.1: Note that if the hypothesis (H_2) hold, then there exist constants $K_1 > 0$ and such that:

$$K_1 = \underbrace{sup}_{t\ge 0} \frac{1}{\Gamma(\zeta)} \int_0^t \frac{m_1(s)}{(t-s)^{1-\zeta}}\, ds$$

Theorem 4.1: Suppose that the hypotheses $[(H_1) - (H_5)]$ are hold. Furthermore if $(\|\alpha\| K_1 + \|\beta\|) < 1$ where K_1 are defined remark (4.1), Then the equation (2.1) has a solution in the space $BC(\mathcal{R}_+, \mathcal{R})$ Moreover, solutions of the equation(2.1) are locally attractive on $\mathcal{R}_+$.

Proof: By a solution of the (2.1) we mean a continuous function $x: \mathcal{R}_+ \to \mathcal{R}$ that satisfies (2.1) on $\mathcal{R}_+$.

Let $X = BC(\mathcal{R}_+, \mathcal{R})$ be Banach Algebras of all continuous and bounded real valuedfunction on $\mathcal{R}_+$ with thenorm$\|x\| = \underbrace{sup}_{t\in\mathcal{R}_+} |x(t)|$(4.1)

We show that existence the solution for (2.1) under some suitable conditions on the functions involved in (2.1).

Consider the closed ball $B_r[0]$ in X centered at origin 0 and of radius r, where

$$r = \|F\| \frac{T^{\delta}}{\Gamma(\delta+1)} \|m_1\| \frac{T^{\zeta}}{\Gamma(\zeta+1)} + \|H_k\| \frac{T^{\beta_k+\delta}}{\Gamma(\beta_k+\delta+1)} > 0$$

Let us define theoperators, $\mathcal{C}: X \to X$and $\mathcal{B}: B_r[0] \to X$ by

$$\mathcal{A}x(t) = I^{\delta} f(t, x(\alpha(t)) \quad (4.2)$$

and $$\mathcal{B}x(t) = \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, x(\mu(t)), I^{\rho} x(\tau(t))\right)}{(t-s)^{1-\zeta}} ds \ \forall t \mathcal{R}_+ \quad (4.3)$$

$$\mathcal{C}x(t) = \sum_{k=1}^{n} I^{\beta_k+\delta} h_k(t, x(\gamma(t)) \quad (4.4)$$

In the view of hypotheses (H_1), the mapping $\mathcal{A}$ is well defined and the function $\mathcal{A}x$ is continuous and bounded on$\mathcal{R}_+$. The function $\mathcal{B}x$ is also continuous and bounded in view of hypotheses(H_2).

We shall show that operators $\mathcal{A}$,$\mathcal{B}$and $\mathcal{C}$ satisfy all the conditions of theorem (3.3).This will be achieved in the following series of steps.

Step I: Firstly, we show that $\mathcal{A}$ is Lipschitz on . Let$x, y \in X$ be arbitrary, and then by hypothesis (H_1), we get

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = \left|I^{\delta} f(t, x(\alpha(t)) - I^{\delta} f(t, y(\alpha(t))\right|$$

$$= \left| \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} f(s, x(\alpha(s)) ds - \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} f(s, y(\alpha(s)) ds \right|$$

$$\leq \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} \left| f(s,x(\alpha(s))) - f(s,y(\alpha(s))\right| ds$$

$$\leq \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} l(t) \left| x - y \right| ds$$

$$\leq \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} L \left| x - y \right| ds$$

$$\leq \frac{t^{\delta}}{\Gamma(\delta)\,\delta} L \left| x - y \right| \text{ for all } t \in \mathcal{R}_+$$

Taking supremum over t

$$\leq \frac{T^{\delta}}{\Gamma(\delta+1)} L \|x-y\|$$

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \boldsymbol{L_1} \|x-y\| \text{for all } \; x,y \in X \qquad (4.5)$$

Where

$$L_1 = \frac{T^{\delta}}{\Gamma(\delta+1)} L$$

This shows that $\mathcal{A}$ is Lipchitzianon X with theLipschitz constant $\boldsymbol{L_1}$.

Step II: Now,we show that $\mathcal{C}$ is Lipschitz on$X$. Let$x,y \in X$ be arbitrary, and then by hypothesis (H_1) , we get

$$|\mathcal{C}x(t) - \mathcal{C}y(t)| = \left| \sum_{k=1}^{n} I^{\beta_k+\delta} h_k(t,x(\gamma(t))) - \sum_{k=1}^{n} I^{\beta_k+\delta} h_k(t,y(\gamma(t))) \right|$$

$$\leq \sum_{k=1}^{n} \int_0^t \frac{(t-s)^{\beta_k+\delta-1}}{\Gamma(\beta_k+\delta)} \lambda_k(s) |x(s) - y(s)| ds$$

$$\leq \|x-y\| \sum_{k=1}^{n} \|\lambda_k\| \frac{T^{\beta_k+\delta}}{\Gamma(\beta_k + \delta + 1)}$$

$$|\mathcal{C}x(t) - \mathcal{C}y(t)| \leq \sum_{k=1}^{n} \frac{\|\lambda_k\| T^{\beta_k+\delta}}{\Gamma(\beta_k + \delta + 1)} \|x-y\|$$

This shows that $\mathcal{C}$ is Lipchitzianon X with the Lipschitz constant

$$\frac{\|\lambda_k\| T^{\beta_k+\delta}}{\Gamma(\beta_k + \delta + 1)}$$

Step III: Secondly, To Prove the operator $\mathcal{B}$ is completely continuous operator on$B_r[0]$.

Firstly we show that $\mathcal{B}$ is continuous on $B_r[0]$.

CaseI: Suppose that $t \geq T$ there exist $T > 0$ andlet us fix arbitrary $\varepsilon > 0$ and take $x, y \in B_r[0]$ such that. $\|x-y\| \leq \varepsilon$ Then

$$|(\mathcal{B}x)t - (\mathcal{B}y)t| \leq$$

$$\left| \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, x(\mu(t)), I^\rho x(\tau(t))\right)}{(t-s)^{1-\zeta}} ds - \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, y(\mu(t)), I^\rho y(\tau(t))\right)}{(t-s)^{1-\zeta}} ds \right|$$

$$\leq \frac{1}{\Gamma(\zeta)} \left[\int_0^t \frac{\left|g\left(t, x(\mu(t)), I^\rho x(\tau(t))\right)\right|}{(t-s)^{1-\zeta}} ds + \int_0^t \frac{\left|g\left(t, y(\mu(t)), I^\rho y(\tau(t))\right)\right|}{(t-s)^{1-\zeta}} ds \right]$$

$$\leq \frac{1}{\Gamma(\zeta)} \left[\int_0^t \frac{m_1(s)}{(t-s)^{1-\zeta}} ds + \int_0^t \frac{m_1(s)}{(t-s)^{1-\zeta}} ds \right]$$

$$\leq \frac{2}{\Gamma(\zeta)} \left[\int_0^t \frac{m_1(s)}{(t-s)^{1-\zeta}} ds \right]$$

$$\leq \frac{2\, v(t)}{\Gamma(\zeta)} \tag{4.6}$$

Hence we see that there exists T > 0 such that

$$v\,(t)\ \leq \frac{\varepsilon\Gamma(\zeta)}{2} \text{ for } t > T$$

Since ε is an arbitrary, from (4.6) we derive that $|(\mathcal{B}x)\,t - (\mathcal{B}y)\,t| \leq \varepsilon$ (4.7)

Case II: Further, let us assume that, $t \in [0, T]$ then evaluating similarly as above we obtain the following estimate

$|(\mathcal{B}x)\,t - (\mathcal{B}y)\,t| \leq$

$$\leq \left| \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, x(\mu(t)), I^\rho x(\tau(t))\right)}{(t-s)^{1-\zeta}} ds - \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, y(\mu(t)), I^\rho y(\tau(t))\right)}{(t-s)^{1-\zeta}} ds \right|$$

$$\leq \frac{1}{\Gamma(\zeta)} \left[\int_0^T \frac{\left| g\left(t, x(\mu(t)), I^\rho x(\tau(t))\right) - g\left(t, y(\mu(t)), I^\rho y(\tau(t))\right) \right|}{(t-s)^{1-\zeta}} ds \right]$$

$$\leq \frac{1}{\Gamma(\zeta)} \left[\int_0^T \frac{w_r^T\,(g, \epsilon)}{(t-s)^{\zeta}} ds \right]$$

$$\leq \left[\frac{w_r^T\,(g, \epsilon)}{\Gamma(\zeta)\zeta} T^\zeta ds \right]$$

$$\leq \left[\frac{w_r^T\,(g, \epsilon)}{\Gamma(\zeta + 1)} T^\zeta ds \right] \tag{4.8}$$

Where

$$w_r^T(g,\epsilon) = \sup\{\left|g\left(t,x(\mu(t)),I^\rho x(\tau(t))\right) - g\left(t,y(\mu(t)),I^\rho y(\tau(t))\right)\right| : s \in [0,T], x, y \in [-r,r], |x-y| \le \epsilon\}$$

Therefore, from the uniform continuity of the function $g(t,x,y)$ on the set $[0,T] \times [-r,r]$ .we derive that $w_r^T(g,\epsilon) \to 0$ as $\varepsilon \to 0$

Now combining the case I and II, we conclude that the operator $\mathcal{B}$ is continuous operator on closed ball $B_r[0]$ in to itself.

Step IV: Next we show that $\mathcal{B}$ is compact on $B_r[0]$.

(A) First prove that every sequence $\{\mathcal{B}x_n\}$ in $\mathcal{B}(B_r[0])$ has a uniformly bounded sequence in $\mathcal{B}(B_r[0])$. Now by $(H_1)-(H_5)$

$$|(\mathcal{B}x_n)t| = \left|\frac{1}{\Gamma(\zeta)}\int_0^t \frac{g\left(t,x(\mu(t)),I^\rho x(\tau(t))\right)}{(t-s)^{1-\zeta}}ds\right|$$

$$|(\mathcal{B}x_n)t| \le \frac{1}{\Gamma(\zeta)}\int_0^t \frac{\left|g\left(t,x(\mu(t)),I^\rho x(\tau(t))\right)\right|}{(t-s)^{1-\zeta}}ds$$

$$|(\mathcal{B}x_n)t| \le \frac{1}{\Gamma(\zeta)}\int_0^t \frac{m_1(s)}{(t-s)^{1-\zeta}}ds$$

$$|(\mathcal{B}x_n)t| \le \frac{v(t)}{\Gamma(\zeta)}$$

$$|(\mathcal{B}x_n)t| \le K_1 \forall t \in \mathcal{R}_+ \qquad (4.9)$$

Taking supremum over t, we obtain $\|\mathcal{B}x_n\| \le K_1 \forall n \in N$

This shows that $\{\mathcal{B}x_n\}$ is a uniformly bounded sequence in $\mathcal{B}(B_r[0])$.

(B) Now we proceed to show that sequence $\{\mathcal{B}x_n\}$ is also equicontinuous.

Let $\varepsilon > 0$ b be given. Since there is constant $T > 0$

Case I: If t_1 , $t_2 \in [0,T]$ then we have

$|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1|$

$$\leq \left|\frac{1}{\Gamma(\zeta)}\int_0^{t_2}\frac{g\left(t, x_n(\mu(t)), I^\rho x_n(\tau(t))\right)}{(t_2-s)^{1-\zeta}}ds - \frac{1}{\Gamma(\beta)}\int_0^{t_1}\frac{g\left(t, x_n(\mu(t)), I^\rho x_n(\tau(t))\right)}{(t_1-s)^{1-\zeta}}ds\right|$$

$$\leq \left|\frac{1}{\Gamma(\zeta)}\int_0^{t_2}\frac{\left|g\left(t, x_n(\mu(t)), I^\rho x_n(\tau(t))\right)\right|}{(t_2-s)^{1-\zeta}}ds - \frac{1}{\Gamma(\zeta)}\int_0^{t_1}\frac{\left|g\left(t, x_n(\mu(t)), I^\rho x_n(\tau(t))\right)\right|}{(t_1-s)^{1-\zeta}}ds\right|$$

$$\leq \left|\frac{1}{\Gamma(\zeta)}\int_0^{t_2}\frac{m_1(s)}{(t_2-s)^{1-\zeta}}ds - \frac{1}{\Gamma(\beta)}\int_0^{t_1}\frac{m_1(s)}{(t_1-s)^{1-\zeta}}ds\right|$$

$$\leq \frac{1}{\Gamma(\zeta)}\left|\int_0^{t_2}\frac{m_1(s)}{(t_2-s)^{1-\zeta}}ds - \int_0^{t_1}\frac{m_1(s)}{(t_1-s)^{1-\zeta}}ds\right|$$

$$\leq \frac{1}{\Gamma(\zeta)}|v(t_2) - v(t_1)| \tag{4.10}$$

from the uniform continuity of the function $v(t)$ on $[0,T]$, we get $|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1| \to 0$ as $t_1 \to t_2$

Case II: If $t_1, \ t_2 \geq T$ then we have

$|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1|$

$$\leq \left|\frac{1}{\Gamma(\zeta)}\int_0^{t_2}\frac{g\left(t, x_n(\mu(t)), I^\rho x_n(\tau(t))\right)}{(t_2-s)^{1-\zeta}}ds - \frac{1}{\Gamma(\zeta)}\int_0^{t_1}\frac{g\left(t, x_n(\mu(t)), I^\rho x_n(\tau(t))\right)}{(t_1-s)^{1-\zeta}}ds\right|$$

$$\leq \left| \frac{1}{\Gamma(\zeta)} \int_0^{t_2} \frac{\left| g\left(t, x_n(\mu(t)), I^{\rho} x_n(\tau(t))\right)\right|}{(t_2 - s)^{1-\zeta}} ds \right.$$

$$\left. - \frac{1}{\Gamma(\zeta)} \int_0^{t_1} \frac{\left| g\left(t, x_n(\mu(t)), I^{\rho} x_n(\tau(t))\right)\right|}{(t_1 - s)^{1-\zeta}} ds \right|$$

$$\leq \left| \frac{1}{\Gamma(\zeta)} \int_0^{t_2} \frac{\left| g\left(t, x_n(\mu(t)), I^{\rho} x_n(\tau(t))\right)\right|}{(t_2 - s)^{1-\zeta}} ds \right|$$

$$+ \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{\left| g\left(t, x_n(\mu(t)), I^{\rho} x_n(\tau(t))\right)\right|}{(t_1 - s)^{1-\zeta}} ds \right|$$

$$\leq \frac{v(t_2)}{\Gamma(\zeta)} + \frac{v(t_1)}{\Gamma(\zeta)} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$\leq \epsilon$ as $t_1 \to t_2$. (4.11)

Case III: If $t_1, t_2 \in \mathcal{R}_+$ With $t_1 < T < t_2$ then we have

$|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1| \leq |\mathcal{B}x_n(t_2) - \mathcal{B}x_n(T)| + |\mathcal{B}x_n(T) - \mathcal{B}x_n(t_1)|$ (4.12)

Now if $t_1 \to t_2$ then $t_1 \to T$ and $T \to t_2$

Therefore, $|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(T)| \to 0$, $|\mathcal{B}x_n(T) - \mathcal{B}x_n(t_1)| \to 0$

and so $|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1| \to 0$ as $t_1 \to t_2$ for all $t_1, t_2 \in \mathcal{R}_+$ (4.13)

Hence $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in $\mathcal{B}(B_r[0])$.

Therefore, it follows from Arzela-Ascoli theorem $\mathcal{B}$ is completely continuous on $B_r[0]$.

Step V: Next we show that $\mathcal{A}x.\mathcal{B}x + \mathcal{C}x \in B_r[0]$ for all $x \in B_r[0]$ is arbitrary, then

$$x(t) = I^{\delta} f(t, x(\alpha(t))) \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, x(\mu(t)), I^{\rho} x(\tau(t))\right)}{(t-s)^{1-\zeta}} ds + \sum_{k=1}^{n} I^{\beta_k+\delta} h_k(t, x(\gamma(t)))$$

$$|\mathcal{A}x(t).\mathcal{B}x(t) + \mathcal{C}x| \le |\mathcal{A}x(t)|.|\mathcal{B}x(t)| + |\mathcal{C}x(t)|$$

$$\le \left|I^{\delta} f(t, x(\alpha(t)))\right| \left|\frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, x(\mu(t)), I^{\rho} x(\tau(t))\right)}{(t-s)^{1-\zeta}} ds\right| + \left|\sum_{k=1}^{n} I^{\beta_k+\delta} h_k(t, x(\gamma(t)))\right|$$

$$\le \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} \left| f(t, x(\alpha(t)))\right| \left[\frac{1}{\Gamma(\zeta)} \int_0^t \frac{\left|g\left(t, x(\mu(t)), I^{\rho} x(\tau(t))\right)\right|}{(t-s)^{1-\zeta}} ds\right]$$

$$+ \sum_{k=1}^{n} \int_0^t \frac{(t-s)^{\beta_k+\delta-1}}{\Gamma(\beta_k+\delta)} \left(\left|h_k(s, x(\gamma(s)))\right|\right) ds$$

$$\le \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} | F(t) | ds \left[\frac{1}{\Gamma(\zeta)} \int_0^t \frac{1}{(t-s)^{1-\zeta}} |m_1(t)| \, ds\right]$$

$$+ \sum_{k=1}^{n} \int_0^t \frac{(t-s)^{\beta_k+\delta-1}}{\Gamma(\beta_k+\delta)} (|H_k(t)| \, ds$$

$$\le \|F\| \frac{T^{\delta}}{\Gamma(\delta+1)} \|m_1\| \frac{T^{\zeta}}{\Gamma(\zeta+1)} + \|H_k\| \frac{T^{\beta_k+\delta}}{\Gamma(\beta_k+\delta+1)} = r$$

Taking the supremum over t, we obtain $\| \mathcal{A}x\mathcal{B}x + \mathcal{C}x \| \le r$ for all $x \in B_r[0]$

Hence hypothesis (c) of Theorem (2.3) holds.

Also we have $M = \| \mathcal{B}(B_r[0]) \| = \sup \{ \| \mathcal{B}x \| : x \in B_r[0]$

$$= sup\{ \underbrace{sup}_{t\ge 0} \left\{ \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, x(\mu(t)), I^{\rho} x(\tau(t))\right)}{(t-s)^{1-\zeta}} ds \quad x \in B_r[0] \right\}$$

$$= sup\{\underbrace{sup}_{t\geq 0}\left\{\frac{1}{\Gamma(\zeta)}\int_0^t \frac{m_1(s)}{(t-s)^{1-\zeta}}ds \quad x\in B_r[0]\right\}$$

$$\leq \underbrace{sup}_{t\geq 0}\left\{\underbrace{sup}_{t\geq 0}\frac{v(t)}{\Gamma(\zeta)}\right\}\leq K_1$$

and therefore $\xi M+\eta$, we have $(\|\alpha\|K_1+\|\beta\|)<1$, Where $\xi=\|\alpha\|$ and $\eta=\|\beta\|$

Now Appling Dhage'sTheorem [2.3] gives that FQFIDE (2.1) has a solution on $\mathcal{R}_+$.

Step VI: Now for the local attractivity of the solutions for (2.1), let's assume that x and $y$be any two solutions of the (2.1) in $B_r[0]$ defined on $\mathcal{R}_+$. Then we have,

$$|x(t)-y(t)| = \Bigg| I^{\delta}f(t,x(\alpha(t)))\frac{1}{\Gamma(\zeta)}\int_0^t\frac{g\left(t,x(\mu(t)),I^{\rho}x(\tau(t))\right)}{(t-s)^{1-\zeta}}ds$$

$$+\sum_{k=1}^{n} I^{\beta_k+\delta}h_k(t,x(\gamma(t)))$$

$$-I^{\delta}f(t,y(\alpha(t)))\frac{1}{\Gamma(\zeta)}\int_0^t\frac{g\left(t,y(\mu(t)),I^{\rho}y(\tau(t))\right)}{(t-s)^{1-\zeta}}ds$$

$$-\sum_{k=1}^{n} I^{\beta_k+\delta}h_k(t,y(\gamma(t)))\Bigg|$$

$$|x(t)-y(t)| \leq \left|\sum_{k=1}^{n} I^{\beta_k+\delta}h_k(t,x(\gamma(t)))\right| + \left|\sum_{k=1}^{n} I^{\beta_k+\delta}h_k(t,y(\gamma(t)))\right|$$

$$+\Bigg| I^{\delta}f(t,x(\alpha(t)))\frac{1}{\Gamma(\zeta)}\int_0^t\frac{g\left(t,x(\mu(t)),I^{\rho}x(\tau(t))\right)}{(t-s)^{1-\zeta}}ds$$

$$-I^{\delta}f(t,y(\alpha(t)))\frac{1}{\Gamma(\zeta)}\int_0^t\frac{g\left(t,y(\mu(t)),I^{\rho}y(\tau(t))\right)}{(t-s)^{1-\zeta}}ds\Bigg|$$

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$$|x(t) - y(t)| \leq 2\|H_k\| \frac{T^{\beta_k+\delta}}{\Gamma(\beta_k + \delta + 1)} \quad (4.14)$$

For all $t \in \mathcal{R}_+$ .Since and $\lim_{t\to\infty} H_k(t) = 0$ this gives that $\lim_{t\to\infty} \sup|x(t) - y(t)| = 0$. Thus the (2 .1) has a solution and all the solutions are locally attractive on $\mathcal{R}_+$

5. Existence of ExtremalSolution:

In this section we consider the following Definitions and show that given equation (2.1) has Maximal and Minimal solution:

Definition 5.1 : (Chandrabhan) A function $f: \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is called chandrabhan if

i) The function $(x,y) \to f(x,y,z)$ is measurable for each $z \in \mathcal{R}$

ii) The function $z \to f(x,y,z)$ is non-decreasing for almost each $(x,y) \in \mathcal{R}_+$

Definition 5.2: A function $p_1 \in BC(\mathcal{R}_+,\mathcal{R})$ is called a **lower solution** of the FQFIDE (2.1) on $\mathcal{R}_+$ if the function

$t \to \left\{\frac{D^\delta p_1(t) - \sum_{k=1}^n I^{\beta_k} h_k(t,p_1(\gamma(t)))}{f(t,p_1(\alpha(t)))}\right\}$ is continuous absolutely and

$$p_1(t) \leq I^\delta f(t,x(\alpha(t))) \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, x(\mu(t)), I^\rho x(\tau(t))\right)}{(t-s)^{1-\zeta}} ds + \sum_{k=1}^n I^{\beta_k+\delta} h_k(t, x(\gamma(t))) \quad (5.1)$$

Again a function $p_2 \in BC(\mathcal{R}_+,\mathcal{R})$ is called an **upper solution** of the FQFIDE (2.1) on $\mathcal{R}_+$ if the function

$$t \to \left\{\frac{D^\delta p_2(t) - \sum_{k=1}^n I^{\beta_k} h_k(t,p_2(\gamma(t)))}{f(t,p_2(\alpha(t)))}\right\}$$

is continuous absolutely and

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$$p_2(t) \geq I^{\delta} f(t, x(\alpha(t)) \frac{1}{\Gamma(\zeta)} \int_0^t \frac{g\left(t, x(\mu(t)), I^{\rho} x(\tau(t))\right)}{(t-s)^{1-\zeta}} ds$$

$$+ \sum_{k=1}^{n} I^{\beta_k + \delta} h_k(t, x(\gamma(t)) \qquad (5.2)$$

Definition 5.3[10,35]:A closed and non-empty set $\mathcal{K}$ in a Banach Algebra X is called a **cone** if

i. $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$

ii. $\lambda\mathcal{K} \subseteq \mathcal{K}$ for $\lambda \in \mathcal{K}, \lambda \geq 0$

iii. $\{-\mathcal{K}\} \cap \mathcal{K} = 0$ where 0 is the zero element of X.

and is called **positive cone** if

iv. $\mathcal{K} \circ \mathcal{K} \subseteq \mathcal{K}$

And the notation $\circ$ is a multiplication composition in X

We introduce an order relation $\leq$ in X as follows.

Let $x, y \in X$ then $x \leq y$ if and only if $y - x \in \mathcal{K}$. A cone $\mathcal{K}$ is called normal if the norm $\|\cdot\|$ is monotone increasing on $\mathcal{K}$. It is known that if the cone $\mathcal{K}$ is normal in X then every order-bounded set in X is norm-bounded set in X.

Definition 5.4 : A solution x_M of the Integral equation is said to be maximal if for any other solution x to the problem $x(t) \leq x_M(t) \forall t \in \mathcal{R}$

Again a solution x_m of the Integral equation is said to be Integral equation if for any other solution x to the problem $x_m(t) \leq x(t) \forall t \in \mathcal{R}$

Lemma 5.1[13]: Let $p_1, p_2, q_1, q_2 \in \mathcal{K}$ be such that $p_1 \leq q_1$and $p_2 \leq q_2$then $p_1 p_2 \leq q_1 q_2$.

For any $p_1, p_2 \in X = \mathcal{C}(\mathcal{R}_+, \mathcal{R})$,$p_1 \leq p_2$ the order interval $[p_1, p_2]$ is a set in$X$given by,
$[p_1, p_2] = \{x \in X : p_1 \leq x \leq p_2\}$

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Definition 5.5[6]: A mapping $R:[p_1,p_2] \to X$ is said to be nondecreasing or monotone increasing if $x \leq y$ implies $Rx \leq Ry$ for all $x, y \in [p_1,p_2]$.

Theorem 5.1[14]: Let $\mathcal{K}$ be a cone in Banach Algebra X and let $[p_1,p_2] \in X$. Suppose that $\mathcal{A}, \mathcal{B}:[p_1,p_2] \to \mathcal{K}$ and $\mathcal{C}:[p_1,p_2] \to X$ be three nondecreasing operators such that

a. $\mathcal{A}$ and $\mathcal{C}$ are a Lipschitz with Lipschitz constant α, β
b. $\mathcal{B}$ is completely continuous,
c. The elements $p_1,p_2 \in X$ satisfy $p_1 \leq \mathcal{A}p_1\mathcal{B}p_1 + \mathcal{C}p_1$ and $\mathcal{A}p_2\mathcal{B}p_2 + \mathcal{C}p_2 \leq p_2$

Further if the cone $\mathcal{K}$ is normal and positive then the operator equation $x = \mathcal{A}x\mathcal{B}y + \mathcal{C}x$ has the least and greatest positive solution in $[p_1,p_2]$ whenever $\alpha M + \beta < 1$, where $M = \|\mathcal{B}([p_1,p_2])\| = sup\{\|\mathcal{B}x\|: x \in [p_1,p_2]\}$.

we consider another hypothesis

(H_6) The function $x \to \left\{\frac{D^\delta x(t) - \sum_{k=1}^{n} I^{\beta_k} h_k(t,x(\gamma(t)))}{f(t,x(\alpha(t)))}\right\}$ is increasing in the interval $\left[min_{t\in\mathcal{R}_+} p_1(t), max_{t\in\mathcal{R}_+} p_2(t)\right]$.

(H_7) The FQFIDE (2.1) has a lower solution p_1 and upper solution p_2 on $\mathcal{R}_+$ with $p_1 \leq p_2$.

(H_8) The function g is caratheodory.

(H_9) The functions $f:\mathcal{R}_+ \times \mathcal{R} \to \mathcal{R} - \{0\}, g:\mathcal{R}_+ \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ and $h_k:\mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ are nondecreasing in x almost everywhere for $t \in \mathcal{R}_+$.

(H_{10}) The function $m_1:\mathcal{R}_+ \to \mathcal{R}$ defined by

$$m_1(t) = \left|g\left(t,p_1(\mu(t)), I^\rho p_1(\tau(t))\right)\right| + \left|g\left(t,p_2(\mu(t)), I^\rho p_2(\tau(t))\right)\right|$$

is Lebesgue measurable.

Remark 5.1: Assume that the hypotheses $((H_6)$ - $(H_{10}))$ holds, then the function $t \to g\left(t,x(\mu(t)), I^\rho x(\tau(t))\right)$ is lebesgue integrable on $\mathcal{R}_+$, say

$\left|g\left(t,x(\mu(t)),I^{\rho}x(\tau(t))\right)\right| \leq m_1(t)$, a.e.,$t \in \mathcal{R}_+$

For all $x \in [p_1,p_2]$ and some Lebesgue integrable function m_1.

Theo 5.2 : Suppose that the Hypothesis((H_6) - (H_{10})) are holds and $\|\alpha\| \left\{\frac{1}{\Gamma(\zeta+1)} T^{\zeta} \|m_1\|_{\mathcal{L}^1}\right\} + \|\beta\| < 1$.Thenproblem (2.1) hasa minimal and maximal positive solutions on $\mathcal{R}$.

Proof : Let $X = C(\mathcal{R}_+,\mathcal{R})$ and we define an order relation "$\leq$" by the cone $\mathcal{K}$ given by (5.3). Clearly $\mathcal{K}$ is a normal cone in X. Define three operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ on X by (4.2), (4.3) and (4.4) respectively. Then FQFIDE (2.1) is transformed into an operator equation $\mathcal{A}x\mathcal{B}x + \mathcal{C}x = x$in Banach algebra X. Notice that (H_6) implies $\mathcal{A},\mathcal{B}:[p_1,p_2] \to \mathcal{K}$ also note that (H_7) ensures that $p_1 \leq \mathcal{A}p_1\mathcal{B}p_1 + \mathcal{C}p_1$ and $\mathcal{A}p_2\mathcal{B}p_2 + \mathcal{C}p_2 \leq p_2$. Since the cone $\mathcal{K}$ in X is normal, $[p_1,p_2]$is a norm bounded set in X. Now it is shown, as in the proof of Theorem (2.1), that $\mathcal{A}$ and $\mathcal{C}$ are Lipschitz with a Lipschitz constant $\|\alpha\|$ and $\|\beta\|$ respectively. Similarly $\mathcal{B}$ is completely continuous operator on $[p_1,p_2]$. Again the hypothesis (H_9) implies that $\mathcal{A}$,$\mathcal{B}$and $\mathcal{C}$ are non-decreasing on $[p_1,p_2]$. To see this, let $x_1,x_2 \in [p_1,p_2]$ be such that $x_1 \leq x_2$. Then by (H_9),

$$x(t) = I^{\delta}f(t,x(\alpha(t))\frac{1}{\Gamma(\zeta)}\int_0^t \frac{g\left(t,x(\mu(t)),I^{\rho}x(\tau(t))\right)}{(t-s)^{1-\zeta}}ds + \sum_{k=1}^{n} I^{\beta_k+\delta}h_k(t,x(\gamma(t))$$

$\mathcal{A}x_1(\mathrm{t}) = I^{\delta}f(t,x_1(\alpha(t)) \leq I^{\delta}f(t,x_2(\alpha(t)) = \mathcal{A}x_2(t)$ for all $\mathrm{t} \in \mathcal{R}_+$

and

$$\mathcal{B}x_1(\mathrm{t}) = \frac{1}{\Gamma(\zeta)}\int_0^t \frac{g\left(t,x_1(\mu(t)),I^{\rho}x_1(\tau(t))\right)}{(t-s)^{1-\zeta}}ds$$

$$\leq \left[\frac{1}{\Gamma(\zeta)}\int_0^t \frac{g\left(t, x_2(\mu(t)), I^{\rho}x_2(\tau(t))\right)}{(t-s)^{1-\zeta}} ds\right] = \mathcal{B}x_2\,(t)$$

$$\mathcal{C}x_1(t) = \sum_{k=1}^{n} I^{\beta_k+\delta}h_k\big(t, x_1(\gamma(t))\big) \leq \sum_{k=1}^{n} I^{\beta_k+\delta}h_k\big(t, x_2(\gamma(t))\big) \leq \mathcal{C}x_2(t), \qquad t \in \mathcal{R}_+$$

So, $\mathcal{B}$ *and* $\mathcal{C}$ are non decreasing operatorson$[x_1, x_2]$

Again by Hypothesis (H_7)

$$p_1(t) \leq I^{\delta}f(t, p_1(\alpha(t))\frac{1}{\Gamma(\zeta)}\int_0^t \frac{g\left(t, p_1(\mu(t)), I^{\rho}p_1(\tau(t))\right)}{(t-s)^{1-\zeta}} ds$$

$$+ \sum_{k=1}^{n} I^{\beta_k+\delta}h_k\big(t, p_1(\gamma(t))\big)$$

$$\leq I^{\delta}f(t, x(\alpha(t))\frac{1}{\Gamma(\zeta)}\int_0^t \frac{g\left(t, x(\mu(t)), I^{\rho}x(\tau(t))\right)}{(t-s)^{1-\zeta}} ds + \sum_{k=1}^{n} I^{\beta_k+\delta}h_k\big(t, x(\gamma(t))\big)$$

$$\leq I^{\delta}f(t, p_2(\alpha(t))\frac{1}{\Gamma(\zeta)}\int_0^t \frac{g\left(t, p_2(\mu(t)), I^{\rho}p_2(\tau(t))\right)}{(t-s)^{1-\zeta}} ds + \sum_{k=1}^{n} I^{\beta_k+\delta}h_k\big(t, p_2(\gamma(t))\big)$$

$\leq p_2(t), \forall t \in \mathcal{R}_+$ and $x \in [p_1, p_2]$

As a result $p_1(t) \leq \mathcal{A}x(t)\mathcal{B}x(t) + \mathcal{C}x(t) \leq p_2(t), \forall t \in \mathcal{R}_+$ and $x \in [p_1, p_2]$

Hence $\mathcal{A}x\mathcal{B}x + \mathcal{C}x \in [p_1, p_2] \forall x \in [p_1, p_2]$

Again $M = \|\mathcal{B}([p_1, p_2])\| = sup\{\|\mathcal{B}x\| : x \in [p_1, p_2]\}$

$$\leq sup\left\{sup_{t\in\mathcal{R}_+}\left\{\frac{1}{\Gamma(\zeta)}\int_0^t \frac{\left|g\left(t, x(\mu(t)), I^{\rho}x(\tau(t))\right)\right|}{(t-s)^{1-\zeta}}\right\} : x \in [p_1, p_2]\right\}$$

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$$\leq sup\left\{\frac{1}{\Gamma(\zeta)}\left[\frac{(t-s)^{\zeta}}{\zeta}\right]_0^t \|m_1\|_{\mathcal{L}^1}\right\} \leq \frac{1}{\Gamma(\zeta+1)}T^{\zeta}\|m_1\|_{\mathcal{L}^1}$$

Since $\alpha M + \beta < \|\alpha\| \left\{\frac{1}{\Gamma(\zeta+1)} T^{\zeta}\|m_1\|_{\mathcal{L}^1}\right\} + \|\beta\| < 1$

Thus by theorem (5.2) given fractional order nonlinrar functional integro-differential equation (2.1) has a minimal and maximal positive solutions on $\mathcal{R}$.

6 Example: Consider the following fractional order quadratic functional ID equation of type (2.1)

$$D^{1/2}\left[\frac{D^{2/3}x(t) - \sum_{k=1}^{3} I^{\beta_k}h_k(t,x(\gamma(t)))}{f\left(t,x(\alpha(t))\right)}\right] = g\left(t,x(\mu(t)),I^{\rho}x(\tau(t))\right)\forall t \in \mathcal{R}_+ \quad (6.1)$$

$x(0)= 0$ and $D^{2/3}x(0)= 0$

$$f\left(t,x(\alpha(t))\right) = (\sin(\pi t + 2t))\left\{\frac{|x(t)|-2}{|x(t)|+5} + \frac{t-8}{15}\right\}$$

$$g\left(t,x(\mu(t)),I^{\rho}x(\tau(t))\right) = \frac{t}{\frac{1}{4}e^t \sin\left\{\frac{4|x|}{2+|x|}\cos\frac{\pi t}{8}\right\} - \frac{1}{4}e^t \sin\left\{\frac{4I^{11/3}|x|}{2+I^{11/3}|x|}\cos\frac{\pi t}{8}\right\}}$$

$$\sum_{k=1}^{3} I^{\beta_k}h_k(t,x(\gamma(t))) = I^{1/4}\left(2+\frac{e^t \sin 4t}{1+|x(t)|}\right) + I^{1/7}\left(\frac{\sin x(t)\cos t}{t+2}+\frac{1}{t}\right) + I^{7/4}\left(\frac{|x(t)|te^t}{1+e^t}\right)$$

$Here\ \zeta = \frac{1}{2}, \delta = \frac{2}{3}\ k = 3\ \beta_1 = \frac{1}{4}, \beta_2 = \frac{1}{7}, \beta_3 = \frac{7}{4}, \rho = \frac{11}{3}, \alpha = \pi t + 2t, \mu = t, \tau = \pi t$

obiviously $\alpha = \pi t + 2t$, $\mu = t$, $\tau = \pi t$ are continuous.

(***a***) First to show hypothesis ($\boldsymbol{H_1}$) satisfied.

$$\left|f\left(t,x(\alpha(t))\right) - f\left(t,y(\alpha(t))\right)\right|$$

$$= \left|\left\{\sin(\pi+2)t\left\{\frac{|x(t)|-2}{|x(t)|+5} + \frac{t-8}{15}\right\} - \left\{\sin(\pi+2)t\left\{\frac{|y(t)|-2}{|y(t)|+5} + \frac{t-8}{15}\right\}\right\}\right\}\right|$$

$$= \left|\sin(\pi+2)t\left\{\frac{|x(t)|-2}{|x(t)|+5} - \frac{|y(t)|-2}{|y(t)|+5}\right\}\right|$$

$$\leq (\sin\pi t + t)|x(t) - y(t)|$$

$$l(t) = \sin(\pi+2)t$$

(***b***) To show that hypothesis ($\boldsymbol{H_2}$) is satisfied:

Here we taking $m_1(t) = {t^3}/{e^{t}}_{14}$ and

$$g\left(t,x(\mu(t)),I^{\rho}x(\tau(t))\right) = \frac{t}{\frac{1}{4}e^t \sin\left\{\frac{4|x|}{2+|x|}\cos\frac{\pi t}{8}\right\} - \frac{1}{4}e^t \sin\left\{\frac{4I^{11/3}|x|}{2+I^{11/3}|x|}\cos\frac{\pi t}{8}\right\}}$$

Implies that $g\left(t,x(\mu(t)),I^{\rho}x(\tau(t))\right) \leq m_1(t)$

$$i.e.\ \frac{t}{\frac{1}{4}e^t\sin\left\{\frac{4|x|}{2+|x|}\cos\frac{\pi t}{8}\right\} - \frac{1}{4}e^t\sin\left\{\frac{4I^{11/3}|x|}{2+I^{11/3}|x|}\cos\frac{\pi t}{8}\right\}} \leq {t^3}/{e^{t}}_{14}$$

Hence hypothesis ($\boldsymbol{H_2}$) satisfied

(***c***) To show that hypothesis ($\boldsymbol{H_4}$) is satisfied:

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$$\sum_{k=1}^{3} I^{\beta_k} h_k(t,x(\gamma(t)))$$

$$= I^{1/4}\left(2+\frac{e^{-t}sin4t}{1+|x(t)|}\right)+I^{1/7}\left(\frac{sint\,.x(t)\,cost}{t+2}+\frac{1}{t}\right)+I^{7/4}\left(\frac{|x(t)|te^{t}}{1+e^{2t}}\right)$$

$$\left|h_1\left((t,x(\gamma(t))\right)-h_1\left(t,y(\gamma(t))\right)\right| = \left|I^{1/4}\left(2+\frac{e^{-t}sin4t}{1+|x(t)|}\right)-I^{1/4}\left(2+\frac{e^{-t}sin4t}{1+|y(t)|}\right)\right|$$

$$\leq e^{-t}sin4t\left|\frac{1+|y(t)|-1-|x(t)|}{1+|x(t)||y(t)|+|x(t)|+|y(t)|}\right|$$

$$\leq e^{-t}sin4t|x(t)-y(t)|$$

It follows that

$$\lambda_1(t) = e^{-t}sin4t$$

$$\left|h_2\left((t,x(\gamma(t))\right)-h_2\left((t,y(\gamma(t))\right)\right|$$

$$= \left|I^{1/7}\left(\frac{sint.x(t)\,cost}{t+2}+\frac{1}{t}\right)--I^{1/7}\left(\frac{sint.y(t)\,cost}{t+2}+\frac{1}{t}\right)\right|$$

$$\leq\left(\frac{sintcost}{t+2}\right)\{|x(t)-y(t)|\}$$

$$\leq\left(\frac{sintcost}{t+2}\right)|x(t)-y(t)|$$

It follows that

$$\lambda_2(t) = \left(\frac{sintcost}{t+2}\right)$$

$$\left|h_3\left((t,x(\gamma(t))\right)-h_3\left((t,x(\gamma(t))\right)\right| = \left|I^{7/4}\left[\left(\frac{|x(t)|te^{t}}{1+e^{2t}}\right)\right]-I^{7/4}\left[\left(\frac{|y(t)|te^{t}}{1+e^{2t}}\right)\right]\right|$$

$$\leq\left(\frac{te^t}{1+e^{2t}}\right)\{||x(t)|-|y(t)||\}$$

It follows that

$$\lambda_3(t)=\left(\frac{te^t}{1+e^{2t}}\right)$$

(***d***)To show that hypothesis(H_3) is satisfied:

$$v(t)=\int_0^t\frac{m_1(t)}{(t-s)^{1-\zeta}}ds=\int_0^t\frac{t^3/e^t14}{(t-s)^{1-\zeta}}ds=\frac{t^3}{e^t14}\int_0^t(t-s)^{\zeta-1}ds$$

$$=\frac{t^3}{e^t14}\left\{\left[\frac{(t-s)^\zeta}{-\zeta}\right]_0^t\right\}$$

$$=\frac{t^3}{e^t14}\left\{\left[\frac{(t-t)^\zeta}{-\zeta}-\frac{(t-0)^\zeta}{-\zeta}\right]\right\}$$

$$=\frac{t^3}{e^t14}\left[\frac{(t)^\zeta}{\zeta}\right]=\frac{t^{\zeta+3}}{e^t14(\xi)}$$

Implies that $v(t)$ is bounded for $t\in\mathcal{R}_+$

Hence the entire hypotheses are satisfied. Consequently all the conditions of theorem (2.3) are satisfied.

Thus problem (6.1) has at least one solution on $t\in\mathcal{R}_+$

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