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Existence Result for Second Order Nonlinear Quadratic Functional Differential Equation

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Abstract: In this paper, we discuss the existence result for second order nonlinear quadratic functional differential equation in $\mathcal{R}_+$ by using hybrid fixed point theorem due to B.C.Dhage. Also locally attractively results and external solutions for fractional order quadratic functional differential equation.

Keywords: Banach algebras, hybrid fixed point theorem, Quadratic functional differential equation, existence result, locally attractive solution, Extremal solution.

2.1 Introduction:

The nonlinear differential and integral equations have been studied extensively in the Literature by several authors for various aspects of the solutions. Fractional order differential and integral equations play a very important role in many applications of real word problem. The study of nonlinear fractional differential and integral equations had been made extensively in the literature by several authors all over the world and now it has become the core part of the nonlinear analysis. The development of nonlinear fractional differential and integral equations though vast growing topic in the subject of nonlinear differential and integral functions [26-30].

Differential and integral equations are one of the most useful Mathematical tools in both applied and pure mathematics. Moreover the theory of Differential and Integral equations is rapidly developing using the tools of Topology, Functional Analysis and Fixed point theory. This is particularly true for problems in the related fields of Engineering, Mechanical Vibrations and Mathematical Physics. There are numerous applications of differential and integral equations of integer and fractional orders in Electrochemistry, Viscoelasticity, Control theory, Electromagnetism and Porous media etc. [7-14, 35-39]

In this paper we will study the existence the solution of second order nonlinear quadratic functional differential equation. The result has been obtained by using hybrid fixed point theorem for two operators in Banach space due to Dhage.

We consider the following second order nonlinear quadratic functional differential equations:

$$\left.\begin{aligned} \mathcal{D}^2\left[\frac{x(t)}{f\left(t,x(\alpha(t))\right)}\right] &= g[t,x(\mu(t))], \qquad t \in \mathcal{R}_+ \\ x(0) &= 0 \end{aligned}\right\} \qquad (2.1.1)$$

Where, $f(t,x):\mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}-\{0\}$, $g(t,x):\mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ and $\alpha,\mu:\mathcal{R}_+ \to \mathcal{R}$

Here the solution of nonlinear differential equations (2.1.1) we mean a function $x \in BC(\mathcal{R}_+,\mathcal{R})$ such that:

(i) The function $t \to \left[\frac{x(t)}{f(t,x(\alpha(t)))}\right]$ is bounded and continuous for each $x \in \mathcal{R}$.

(ii) x satisfies (2.1.1)

2.2 Preliminaries

This section is devoted to collecting the definitions, notation, hypothesis and preliminary tools, which will be used in the sequel.

Let $X = BC(\mathcal{R}_+,\mathcal{R})$ be the space of bounded real valued continuous function on $\mathcal{R}_+$ and S be a subset of X. Let a mapping $\mathcal{A}:X \to X$ be an operator and consider the following operator equation in X, namely,

$$x(t) = (\mathcal{A}x)(t), \text{ for all } t \in \mathcal{R}_+ \qquad (2.2.1)$$

Following are some different characterization of the solutions for operator equation (2.2.1) on $\mathcal{R}_+$. We require the following definitions.

Definition 2.2.1[34]: Let (X, d) be the metric space and $a \in X$ and for some real number $r > 0$ the set $B_r[a] = \{x \in X: d(x,a) \le r\}$ is called closed ball centered at a with radius r.

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Definition 2.2.2[22]: We say that solution of the equation (2.2.1) is locally attractive if there exists a closed ball $B_r[0]$ in the space $BC(\mathcal{R}_+,\mathcal{R})$ for some $x_0 \in BC(\mathcal{R}_+,\mathcal{R})$ and for some real number $r > 0$ such that for arbitrary solution $x = x(t)$ and $y = y(t)$ of equation (2.2.1) belonging to $B_r[0] \cap S$ we have that, $\lim_{t\to\infty}(x(t) - y(t)) = 0$ (2.2.2)

Definition 2.2.3[22]: Let X be a Banach space. A mapping $\mathcal{A}: X \to X$ is called Lipschitz if there is a constant $\alpha > 0$ such that, $\|\mathcal{A}x - \mathcal{A}y\| \le \alpha\|x - y\|$ for all $x, y \in X$. If $\alpha < 1$, then $\mathcal{A}$ is called a contraction on X with the contraction constant α.

Definition 2.2.4[18]:.An operator $\mathcal{U}$ on a Banach space X into itself is called compact if for any bounded subset S of X, $\mathcal{U}(S)$ is relatively compact subset of X. If $\mathcal{U}$ is continuous and compact, then it is called completely continuous on X.

Definition 2.2.5[18]: Let X be a Banach space with the norm $\|\cdot\|$ and let $\mathcal{U}: X \to X$ be an operator (in general nonlinear). Then $\mathcal{U}$ is called

i. Compact if $\mathcal{U}(X)$ is relatively compact subset of X.
ii. Totally bounded if $\mathcal{U}(S)$ is totally bounded subset of X for any bounded subset S of X.
iii. Completely continuous if it is continuous and totally bounded operator on X

It is clear that every compact operator is totally bounded but the converse need not be true.

Definition 2.2.6[21]: Let $f \in \mathcal{L}^1[0,\mathcal{T}]$ and $\alpha > 0$. The Riemann – Liouville fractional derivative of order ζ of real function f is defined as

$$\mathcal{D}^{\zeta} f(t) = \frac{1}{\Gamma(1-\zeta)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^{\zeta}} ds \quad , \quad 0 < \zeta < 1$$

Such that $\mathcal{D}^{-\zeta} f(t) = I^{\zeta} f(t) = \frac{1}{\Gamma(\zeta)} \int_0^t \frac{f(s)}{(t-s)^{1-\zeta}} ds$ respectively.

Definition 2.2.6.1 [21]: The Riemann-Liouville fractional integral of order $\zeta \in (0,1)$ of the function $f \in \mathcal{L}^1[0,\mathcal{T}]$ is defined by the formula:

$$I^{\zeta} f(t) = \frac{1}{\Gamma(\zeta)} \int_0^t \frac{f(s)}{(t-s)^{1-\zeta}} ds \, , \quad t \in [0,T]$$

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Where $\Gamma(\zeta)$ denote the Euler gamma function. The Riemann-Liouville fractional derivative operator of order ζ defined by

$$\mathcal{D}^{\zeta} = \frac{d^{\zeta}}{dt^{\zeta}} = \frac{d}{dt} \circ I^{1-\zeta}$$

It may be shown that the fractional integral operator I^{ζ} transforms the space $\mathcal{L}^1(\mathcal{R}_+,\mathcal{R})$ into itself and has some other properties.

Definition 2.2.7 [34]:A subset A of a metric space (*X*,d)is said to be relatively compact if $\bar{A}$ is compact.

Definition 2.2.8 [34]:$\Gamma(n+1) = n!$

Theorem 2.2.1 [6] :(Arzela-Ascoli Theorem) If every uniformly bounded and equicontinuous sequence $\{f_n\}$ of functions in $\mathcal{C}(\mathcal{R}_+,\mathcal{R})$, then it has a convergent subsequence.

Theorem 2.2.2[6]: A metric space X is compact iff every sequence in X has a convergent subsequence.

Theorem 2.2.3[5, 6, and 17]: Let S be a non-empty, bounded and closed-convex subset of the Banach space X and let $\mathcal{A}: X \to X$ and $\mathcal{B}: S \to X$ are two operators satisfying

a) $\mathcal{A}$ is Lipschitz with a lipschitz constant α,
b) $\mathcal{B}$ is completely continuous, and
c) $\mathcal{A}x\mathcal{B}x \in S$ for all $x \in S$, and
d) $\alpha M < 1$,where $M = \|\mathcal{B}(S)\|: \sup\{\|\mathcal{B}x\| : x \in S\}$.

Then the operator equation $\mathcal{A}x\mathcal{B}x = x$ has a solution in S.

Remark: we have seen that compact sets are always closed, so we can say that this compact sets are relatively compact.

2.3 Existence Theory

Now we want the solution of (2.2.1) in the space $BC(\mathcal{R}_+,\mathcal{R})$of bounded and continuous real valued functions defined on $\mathcal{R}_+$. Define a standard norm $\|\cdot\|$ and a multiplication " . " in $BC(\mathcal{R}_+,\mathcal{R})$by,$\|x\| = sup\{|x(t)| : t \in \mathcal{R}_+\}, \ (xy)(t) = x(t)y(t), \ t \in \mathcal{R}_+$ (2.3.1)

Clearly, $BC(\mathcal{R}_+,\mathcal{R})$ becomes a Banach space with respect to the above norm and the multiplication in it. By $\mathcal{L}^1(\mathcal{R}_+,\mathcal{R})$ we denote the space of Lebesgue-integrable function in$\mathcal{R}_+$with the norm $\|\cdot\|_{\mathcal{L}^1}$ defined by $\|x\|_{\mathcal{L}} = \int_0^{\infty} |x(t)|\,dt$ (2.3.2)

Definition 2.3.1[6]: A mapping $g:\mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is Caratheodory if:

i) $t \to g(t,x)$ is measurable for each $x \in \mathcal{R}$ and

ii) $x \to g(t,x)$ is continuous almost everywhere for $t \in \mathcal{R}_+$.

Furthermore a Caratheodory function g is $\mathcal{L}^1$ –Caratheodory if:

iii) For each real number $r > 0$ there exists a function $h_r \in \mathcal{L}^1(\mathcal{R}_+,\mathcal{R})$ such that $|g(t,x)| \le h_r(t)\, a.e.\ t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$ with $|x| \le r$

Finally a caratheodory function g is $\mathcal{L}^1_X$ –caratheodory if:

iv) There exists a function $h \in \forall\mathcal{L}^1(\mathcal{R}_+,\mathcal{R})$ such that $|g(t,x)| \le h(t)$, $a.e.\ t \in \mathcal{R}_+$ for all $x \in \mathcal{R}$

For convenience, the function h is referred to as a bound function for g.

2.4 Main Result

We need following hypothesis for existence of solution of second order nonlinear functional differential equation (SNFDE) (2.1.1)

$(\mathcal{H}_5)$ The functions $\alpha,\mu:\mathcal{R}_+ \to \mathcal{R}$ are continuous.

$(\mathcal{H}_6)$The function $f:\mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is continuous and bounded with bound F= $sup_{(t,x(\alpha(t)))\in\mathcal{R}_+\times\mathcal{R}} \left|f\left(t,x(\alpha(t))\right)\right|$there exist a bounded function $l:\mathcal{R}_+ \to \mathcal{R}$ with bound L satisfying

$$\left|f\left(t,x(\alpha(t))\right) - f\left(t,y(\alpha(t))\right)\right|$$
$$\le l(t)\{|x(\alpha(t)) - y(\alpha(t))|\}, a.e.\ t \in \mathcal{R}_+$$

for all $x, y \in \mathcal{R}$.

$(\mathcal{H}_7)$ The function $g:\mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ is satisfying caratheodory condition with continuous function $h(t):\mathcal{R}_+ \to \mathcal{R}$ such that

$g(t,x) \le h(t)\ \forall t \in \mathcal{R}_+$and $x, y \in \mathcal{R}$.

$(\mathcal{H}_8)$ The function $v:\mathcal{R}_+ \to \mathcal{R}$ defined by the formulas $v(t) = \int_0^t (t-s)h(s)\,ds$ is bounded on $\mathcal{R}_+$ and vanish at infinity, that is $\lim_{t\to\infty} v(t) = 0$.

Remark 2.4.1: Note that the $(\mathcal{H}_7)$ and $(\mathcal{H}_8)$ hold, then there exists a constant $K_1 > 0$ such that $K_1 = \sup\{v(t) : t \in \mathcal{R}_+\}$

Lemma 2.4.1: Suppose that $\zeta \in (0,1)$ and the function f, g satisfying SNFDE (2.1.1) then x is the solution of the SNFDE (2.1.1) if and only if it is the solution of integral equation

$$x(t) = \left[f\left(t, x(\alpha(t))\right)\right]\left[\int_0^t (t-s)\, g\left(s, x(\mu(s))\right) ds\right] t \in \mathcal{R}_+ \tag{2.4.1}$$

Proof: Integrating equation (2.1.1) of second order, we get,

$$I\mathcal{D}^2\left[\frac{x(t)}{f\left(t, x(\alpha(t))\right)}\right]_0^t = I\left[g\left(s, x(\mu(s))\right)\right]$$

$$\mathcal{D}\left[\frac{x(t)}{f\left(t, x(\alpha(t))\right)}\right]_0^t = I\left[g\left(s, x(\mu(s))\right)\right]$$

$$\mathcal{D}\left[\frac{x(t)}{f\left(t, x(\alpha(t))\right)}\right] = I\left[g\left(s, x(\mu(s))\right)\right]$$

Again integrating, we get

$$\left[\frac{x(t)}{f\left(t, x(\alpha(t))\right)}\right] = I^2\left[g\left(s, x(\mu(s))\right)\right]$$

$$x(t) = \left[f\left(t, x(\alpha(t))\right)\right]\left[I^2\left[g\left(s, x(\mu(s))\right)\right]\right]$$

$$x(t) = \left[f\left(t, x(\alpha(t))\right)\right]\frac{1}{(2-1)!}\int_0^t (t-s)\, g\left(s, x(\mu(s))\right) ds$$

$$x(t) = \left[f\left(t, x(\alpha(t))\right)\right]\left[\int_0^t (t-s)\, g\left(s, x(\mu(s))\right) ds\right], t \in \mathcal{R}_+$$

Since $\int_0^t f(t)\,dt^n = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)\,ds$, Where $n = 0,1,2,3, \ldots\ldots\ldots$

Conversely differentiate (2.4.1) of order 2 w.r.to t, we get,

$$\mathcal{D}^2\left[\frac{x(t)}{f\left(t,x(\alpha(t))\right)}\right] = \mathcal{D}^2\left[\int_0^t (t-s)\, g\left(s,x(\mu(s))\right) ds\right]$$

$$\mathcal{D}^2\left[\frac{x(t)}{f\left(t,x(\alpha(t))\right)}\right] = \mathcal{D}^2\left[\frac{1}{\Gamma(2)}\int_0^t (t-s)^{2-1} g\left(s,x(\mu(s))\right) ds\right]$$

$$\mathcal{D}^2\left[\frac{x(t)}{f\left(t,x(\alpha(t))\right)}\right] = \quad g\left(s,x(\mu(t))\right)$$

Theorem 2.4.2: Assume that condition $(\mathcal{H}_5)$-$(\mathcal{H}_8)$ hold. Further if $LK_1 < 1$, where K_1 is defined in remark (2.4.1). Then SNFDE (2.1.1) has a solution in the space $BC(\mathcal{R}_+,\mathcal{R})$, moreover solution of (2.1.1) are locally attractive on $\mathcal{R}_+$.

Proof: By a solution of SNFDE (2.1.1) we mean a continuous function $x: \mathcal{R}_+ \to \mathcal{R}$ that satisfies SNFDE (2.1.1) on $\mathcal{R}_+$. Let $X = BC(\mathcal{R}_+,\mathcal{R})$ and define a subset $B_r[0]$ of X as $B_r[0] = \{x \in X: \|x\| \leq r\}$.where r satisfies the inequality, $FK_1 \leq r$.

Let $X = BC(\mathcal{R}_+,\mathcal{R})$ be Banach algebra of all bounded continuous real-valued function on $\mathcal{R}_+$ with the norm $\|x\| = sup|x(t)|, t \in \mathcal{R}_+$ (2.4.2)

Under some suitable conditions involved in (2.1.1) we obtain the solution of SNFDE (2.1.1)

Now the SNFDE (2.1.1) is equivalent to the SNFIE

$$x(t) = \left[f\left(t,x(\alpha(t))\right)\right]\left[\int_0^t (t-s)\, g\left(s,x(\mu(s))\right) ds\right]$$

Let us define the two mappings $\mathcal{A}: X \to X$

and $\mathcal{B}: B_r[0] \to X$ by

$$\mathcal{A}x(t) = f\left(t,x(\alpha(t))\right), t \in \mathcal{R}_+ \qquad (2.4.3)$$

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$$\mathcal{B}x(t) = \int_0^t (t-s)\, g\left(s, x(\mu(s))\right) ds,\ t \in \mathcal{R}_+ \tag{2.4.4}$$

Thus from the SNDE (2.1.1), we obtain the operator equation as follows:

$$x(t) = \mathcal{A}x(t)\mathcal{B}x(t),\ t \in \mathcal{R}_+ \tag{2.4.5}$$

If the operator $\mathcal{A}$ and $\mathcal{B}$ satisfy all the hypothesis of theorem (2.2.3), then the operator equation (2.4.5) has a solution on $B_r[0]$.

Step I: Firstly we show that $\mathcal{A}$ is Lipschitz on$X$ Let $x, y \in X$; then

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| \le \left|f\left(t, x(\alpha(t))\right) - f\left(t, y(\alpha(t))\right)\right|$$

$$\le l(t)\{|x(\alpha(t)) - y(\alpha(t))|\}$$

$$\le L|x(t) - y(t)| \text{for all } t \in \mathcal{R}_+$$

Taking supremum over $t$we get,

$$\|\mathcal{A}x - \mathcal{A}y\| \le L\|x - y\| \text{for all } x, y \in B_r[0]$$

Thus, $\mathcal{A}$ is Lipchitz on X with Lipschitz constant L.

Step II: Secondly we show that $\mathcal{B}$ is completely continuous operator on $B_r[0]$using standard argument such as those in Granas at [18], it can be shown that $\mathcal{B}$ is continuous operator on $B_r[0]$.To do this, let us fix arbitrary $\epsilon > 0$ and take $x, y \in B_r[0]$such that$\|x - y\| \le \epsilon$.

$$|\mathcal{B}x(t) - \mathcal{B}y(t)| = \left| \int_0^t (t-s)\, g\left(s, x(\mu(s))\right) ds - \int_0^t (t-s)\, g\left(s, y(\mu(s))\right) ds \right|$$

$$\le \left|\int_0^t (t-s)\, g\left(s, x(\mu(s))\right) ds\right|$$

$$+ \left|\int_0^t (t-s)\, g\left(s, y(\mu(s))\right) ds\right|$$

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$$\leq \int_0^t (t-s)h(s)\,ds + \int_0^t (t-s)h(s)\,ds$$

$\leq 2\int_0^t (t-s)h(s)\,ds$,

$\leq 2\int_0^t \frac{1}{(t-s)^{1-2}} h(s)\,ds \leq 2v(t)$ (by Hypothesis H_8)

As $v(t) \leq \frac{\epsilon}{2}, |\mathcal{B}x(t) - \mathcal{B}y(t)| \leq \epsilon.$

Thus $\mathcal{B}$ is continuous.

Step III: Now we will show that $\mathcal{B}$ is compact on $\mathcal{B}(B_r[0])$

a) First we prove that every sequence $\{ \mathcal{B}x_n \}$ in $\mathcal{B}(B_r[0])$ has uniformly bounded sequence and $\{ \mathcal{B}x_n \}$ is equicontinuous set in$B_r[0]$. Since $g\left(t, x(\mu(t))\right)$ is $\mathcal{L}_X^1$ – caratheodary, we have

$$|\mathcal{B}x_n(t)| = \left| \int_0^t (t-s)\, g\left(s, x_n(\mu(s))\right) ds \right|$$

$$\leq \int_0^t (t-s) \left| g\left(s, x_n(\mu(s))\right) \right| ds$$

$$\leq \int_0^t (t-s)h(s)\,ds$$

$\leq \int_0^t \frac{1}{(t-s)^{1-2}} h(s)\,ds \leq v(t)$ (by Hypothesis H_8)

Taking supremum over t, we obtain, $\|\mathcal{B}x_n\| \leq K_1$ for all $x \in B_r[0]$

Where, $K_1 = sup_{t\in\mathcal{R}_+}\{v(t)\}$

This shows that $\{ \mathcal{B}x_n \}$ is uniformly bounded sequence in $\mathcal{B}(B_r[0])$

To show that $\{ \mathcal{B}x_n\}$ is an equicontinuous sequence, let $t_1, t_2 \in [0,T]$ be arbitrary. Then for any $x \in B_r[0]$ (2.4.5) implies

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| = \left| \int_0^{t_2} (t_2 - s) g\left(s, x_n(\mu(s))\right) ds - \int_0^{t_1} (t_1 - s) g\left(s, x_n(\mu(s))\right) ds \right|$$

$$= \left| \int_0^{t_2} (t_2 - s) h(s) ds - \int_0^{t_1} (t_1 - s) h(s) ds \right|$$

$$\leq |v(t_2) - v(t_1)|$$

The right hand side of the above inequality doesn't depend on x and tends to zero as $t_1 \to t_2$. Therefore $|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0$ as $t_1 \to t_2$.

If $t_1, t_2 \geq T$ then we have

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| = \left| \int_0^{t_2} (t_2 - s) g\left(s, x_n(\mu(s))\right) ds - \int_0^{t_1} (t_1 - s) g\left(s, x_n(\mu(s))\right) ds \right|$$

$$= \left| \int_0^{t_2} (t_2 - s) h(s) ds - \int_0^{t_1} (t_1 - s) h(s) ds \right|$$

$$\leq \left| \int_0^{t_2} (t_2 - s) h(s) ds \right| + \left| \int_0^{t_1} (t_1 - s) h(s) ds \right| \quad \text{(by Hypothesis } H_8 \text{)}$$

$$\leq v(t_2) + v(t_1) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon \text{ as } t_1 \to t_2$$

If $t_1, t_2 \in \mathcal{R}_+$ then we have

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \leq |\mathcal{B}x_n(t_2) - \mathcal{B}x_n(T)| + |\mathcal{B}x_n(T) - \mathcal{B}x_n(t_1)|$$

If $t_1 \to t_2$, then $t_1 \to T$ and $T \to t_2$

Therefore $|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(T)| \to 0$ $\quad |\mathcal{B}x_n(T) - \mathcal{B}x_n(t_1)| \to 0$

So $|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0$ as $t_1 \to t_2$

Hence, $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in $\mathcal{B}(B_r[0])$ so $\mathcal{B}(B_r[0])$ is relatively compact by Arzela-Ascoli theorem. By definition 2.2.4 $\mathcal{B}$ *is compact* which gives, $\mathcal{B}$ is compact and continuous operator on $B_r[0]$.

Thus $\mathcal{B}$ is completely continuous on $B_r[0]$

Step IV: To show $x = \mathcal{A}x\mathcal{B}y \in B_r[0]$

Let $xy \in B_r[0]$ such that $x = \mathcal{A}x\mathcal{B}x$

By assumptions $(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$

$|x(t)| = |\mathcal{A}x(t)\mathcal{B}x(t)|$

$\leq |\mathcal{A}x(t)||\mathcal{B}x(t)|$

$$\leq \left|f\left(t, x(\alpha(t))\right)\right| \left|\int_0^t (t-s)\, g\left(s, x(\mu(s))\right) ds\right|$$

$$\leq \left|f\left(t, x(\alpha(t))\right)\right| \int_0^t (t-s) \left|g\left(s, x(\mu(s))\right)\right| ds$$

$$\leq F \int_0^t (t-s) h(s)\, ds \leq Fv(t) \text{(by Hypothesis } H_8 \text{)}$$

Taking supremum over ton $\mathcal{R}_+$, we obtain $\|\mathcal{A}x\mathcal{B}x\| \leq FK_1$, $\forall x \in B_r[0]$

That is we have, $\|x\| = \|\mathcal{A}x\mathcal{B}x\| \leq r$, $\forall x \in B_r[0]$.

which gives $x = \mathcal{A}x\mathcal{B}y \in B_r[0]$

Hence assumption(c) of theorem (2.2.3) is proved.

Step V: Also we have

$M = \|\mathcal{B}(B_r[0])\| = sup\{\|\mathcal{B}x\| : x \in (B_r[0])\}$

$$= sup\left\{ sup_{t\in\mathcal{R}_+}\left[\left|\int_0^t (t-s)\,g\left(s,x(\mu(s))\right)ds\right| : x\in B_r[0]\right]\right\}$$

$$\le sup\left\{ sup_{t\in\mathcal{R}_+}\left[\left|\int_0^t (t-s)\,h(s)\,ds\right| : x\in B_r[0]\right]\right\}$$

$$\le sup\{sup_{t\in\mathcal{R}_+}[v(t)] : x\in B_r[0]\}$$

$$\le K_1$$

and therefore $LM = LK_1 < 1$

Thus the condition (d) of theorem (2.2.3) is satisfied.

Hence all the conditions of theorem (2.2.3) are satisfied and therefore the operator equation $\mathcal{A}x\mathcal{B}x = x$ has a solution in $B_r[0]$. As a result, the SNFDE (2.1.1) has a solution defined on $\mathcal{R}_+$.

Step VI: Finally we show the locally attractivity of the solutions for SNFDE (2.1.1). Let x and y be two solutions of SNFDE (2.1.1) in $B_r[0]$ defined on $\mathcal{R}_+$ .Then we have

$$|x(t)-y(t)| = \left| \begin{array}{l} \left[f\left(t,x(\alpha(t))\right)\right]\left[\int_0^t (t-s)\,g\left(s,x(\mu(s))\right)ds\right] - \\ \left[f\left(t,y(\alpha(t))\right)\right]\left[\int_0^t (t-s)\,g\left(s,y(\mu(s))\right)ds\right] \end{array} \right|$$

$$\le \left|\left[f\left(t,x(\alpha(t))\right)\right]\left[\int_0^t (t-s)\,g\left(s,x(\mu(s))\right)ds\right]\right| +$$

$$\left|\left[f\left(t,y(\alpha(t))\right)\right]\left[\int_0^t (t-s)\,g\left(s,y(\mu(s))\right)ds\right]\right|$$

$$\leq \left|f\left(t,x(\alpha(t))\right)\right| \int_0^t (t-s) \left|g\left(s,x(\mu(s))\right)\right| ds +$$

$$\left|f\left(t,y(\alpha(t))\right)\right| \int_0^t (t-s) \left|g\left(s,y(\mu(s))\right)\right| ds$$

$$\leq F\left\{\int_0^t (t-s)h(s)\,ds\right\} + F\left\{\int_0^t (t-s)h(s)\,ds\right\}$$

$$\leq 2F\int_0^t (t-s)h(s)\,ds \leq 2F[v(t)] \qquad \text{(by Hypothesis } H_8 \text{)}$$

Since $\lim_{t\to\infty} v(t) = 0$ for $\epsilon > 0$, there is real number $T > 0$ such that

$v(t) \leq \frac{\epsilon}{2F}$ for all $t \geq T$. Then from above inequality it follows that

$|x(t) - y(t)| < \epsilon$ for all $t \geq T$. This completes the proof.

2.5 Existence of Extremal Solutions:

Definition 2.5.1[10, 36]: A closed and non-empty set $\mathcal{K}$ in a Banach Algebra X is called a cone if

i. $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$
ii. $\lambda\mathcal{K} \subseteq \mathcal{K}$ for $\lambda \in \mathcal{K}, \lambda \geq 0$
iii. $\{-\mathcal{K}\} \cap \mathcal{K} = 0$ where 0 is the zero element of X.

and is called positive cone if

iv. $\mathcal{K} \circ \mathcal{K} \subseteq \mathcal{K}$

and the notation $\circ$ is a multiplication composition in X
We introduce an order relation $\leq$ in X as follows.

Let $x,y \in X$ then $x \leq y$ if and only if $y - x \in \mathcal{K}$. A cone $\mathcal{K}$ is called normal if the norm $\|\cdot\|$ is monotone increasing on $\mathcal{K}$. It is known that if the cone $\mathcal{K}$ is normal in X then every order-bounded set in X is norm-bounded set in X.The details of cone and their properties appear in Guo and Lakshikantham [10]

We equip the space $\mathcal{C}(\mathcal{R}_+,\mathcal{R})$ of continuous real valued function on $\mathcal{R}_+$ with the order relation $\leq$ with the help of cone defined by,

$$\mathcal{K} = \{x \in \mathcal{C}(\mathcal{R}_+,\mathcal{R}): x(t) \geq 0 \ \forall t \in \mathcal{R}_+\} \qquad 1$$

It is well known that the cone $\mathcal{K}$ is normal and positive in $\mathcal{C}(\mathcal{R}_+,\mathcal{R})$. As a result of positivity of the cone $\mathcal{K}$ we have:

Lemma 2.5.1[13]: Let $p_1,p_2,q_1,q_2 \in \mathcal{K}$ be such that $p_1 \leq q_1$ and $p_2 \leq q_2$ then $p_1 p_2 \leq q_1 q_2$.

For any $p_1,p_2 \in X = \mathcal{C}(\mathcal{R}_+,\mathcal{R}), p_1 \leq p_2$ the order interval $[p_1,p_2]$ is a set in X given by, $[p_1,p_2] = \{x \in X: p_1 \leq x \leq p_2\}$

Definition 2.5.2[6]: A mapping $R:[p_1,p_2] \to X$ is said to be nondecreasing or monotone increasing if $x \leq y$ implies $Rx \leq Ry$ for all $x, y \in [p_1,p_2]$.

For proving the existence of extremal solutions of the equations (2.1.1) under certain monotonicity conditions by using following fixed pint theorem of Dhage [9]

Theorem 2.5.1 [14] : Let $\mathcal{K}$ be a cone in Banach Algebra X and let $[p_1,p_2] \in X$. Suppose that $\mathcal{A},\mathcal{B}:[p_1,p_2] \to \mathcal{K}$ be two non-decreasing operators such that

a. $\mathcal{A}$ is Lipschitz with Lipschitz constant α
b. $\mathcal{B}$ is completely continuous,
c. $\mathcal{A}x\mathcal{B}x \in [p,q]$ for each $x \in [p,q]$ and
d. $\mathcal{A}$ and $\mathcal{B}$ are nondecreasing.

Further if the cone $\mathcal{K}$ is normal and positive then the operator equation $x = \mathcal{A}x + \mathcal{B}y$ has the least and greatest positive solution in $[p_1,p_2]$ whenever $\alpha M < 1$, where $M = \|\mathcal{B}([p_1,p_2])\| = sup\{\|\mathcal{B}x\|: x \in [p_1,p_2]\}$

Definition 2.5.3: A function $p_1 \in BC(\mathcal{R}_+,\mathcal{R})$ is called a **lower solution** of the SNFDE (2.1.1) on $\mathcal{R}_+$ if the function $t \to \frac{p_1(t)}{f(t,p_1(\alpha(t)))}$ is continuous and

$$\left.\begin{aligned} \mathcal{D}^2\left[\frac{p_1(t)}{f\left(t,p_1(\alpha(t))\right)}\right] \leq g[t,p_1(\mu(t))], a.e.,t \in \mathcal{R}_+ \\ x(0) = 0 \end{aligned}\right\}$$

Again a function $p_2 \in BC(\mathcal{R}_+,\mathcal{R})$ is called an **upper solution** of the SNFDE (2.1.1) on $\mathcal{R}_+$ ifthe function $t \to \frac{p_2(t)}{f(t,p_2(\alpha(t)))}$ is continuous and

$$\left.\begin{aligned} \mathcal{D}^2\left[\frac{p_2(t)}{f\left(t,p_2(\alpha(t))\right)}\right] \geq g[t,p_2(\mu(t))], a.e.,t \in \mathcal{R}_+ \\ x(0) = 0 \end{aligned}\right\}$$

Definition2.5.4: A solution x_M of the SNFDE (2.1.1) is said to be **maximal** if for any other solution x to SNFDE (2.1.1) one has $x(t) \leq x_M(t)$ for all t $\in \mathcal{R}_+$.Again a solution x_M of the SNFDE (2.1.1) is said to be minimal if $x_M(t) \leq x(t)$ for all t $\in \mathcal{R}_+$ where x is any solution of the SNFDE (2.1.1) on $\mathcal{R}_+$.

We consider the following hypothesis for existence of extremal solution:

$\mathfrak{B}5$) g is Caratheodory.

$\mathfrak{B}6$) The functions $f(t,x(\alpha(t)))$ and $g[t,x(\mu(t))]$ are non-decreasing in x almost everywhere for $t \in \mathcal{R}_+$

$\mathfrak{B}7$) The SNDE (2.1.1) has a lower solution p_1 and an upper solution p_2 on $\mathcal{R}_+$ with $p_1 \leq p_2$

$\mathfrak{B}8$) The function $k:\mathcal{R}_+ \to \mathcal{R}$ defined by $k(t) = |g[t,p_1(\mu(t))]| + |g[t,p_2(\mu(t))]|$is Lebesgue measurable.

Remark 2.5.1: Assume that ($\mathfrak{B}6 - \mathfrak{B}8$) hold. Then

$$|g[t,x(\mu(t))]| \leq k(t), a.e. t \in \mathcal{R}_+$$

for all $x \in [p_1,p_2]$.

Theorem 2.5.2: Suppose that the assumptions ($\mathcal{H}_5$)-($\mathcal{H}_8$) and ($\mathfrak{B}5 - \mathfrak{B}8$) holds and k is given in remark 2.5.1 further if $LT\|k\|_{\mathcal{L}^1} \leq 1$ then SNFDE (2.1.1) has a minimal and maximal positive solution on $\mathcal{R}_+$.

Proof: Now SNFDE (2.1.1) is equivalent to IE (2.4.1) $\mathcal{R}_+$. Let $X = \mathcal{C}(\mathcal{R}_+,\mathcal{R})$ and define an order relation "$\leq$" by the cone $\mathcal{K}$ given by (2.5.1). Clearly $\mathcal{K}$ is a normal cone in X. Define two operators $\mathcal{A}$ and $\mathcal{B}$ on X by (2.4.3) and (2.4.4) respectively. Then IE (2.4.1) is transformed into an operator equation $\mathcal{A}x\mathcal{B}x = x$ in a Banach algebra X. Notice that $(\mathfrak{B}5)$ implies $\mathcal{A},\mathcal{B}: [p_1,p_2] \to \mathcal{K}$ Since the cone $\mathcal{K}$ in X is normal, $[p_1,p_2]$ is a norm bounded set in X. Now it is shown, as in the proof of Theorem (2.4.1), that $\mathcal{A}$ is a Lipschitz with a Lipschitz constant L and $\mathcal{B}$ is completely continuous operator on $[p_1,p_2]$. Again the hypothesis $(\mathfrak{B}6)$ implies that $\mathcal{A}$ and $\mathcal{B}$ are non-decreasing on $[p_1,p_2]$.

To see this, let $x,y \in [p_1,p_2]$ be such that $x \leq y$. Then by $(\mathfrak{B}6)$

$$\mathcal{A}x(t) = f\left(t,x(\alpha(t))\right) \leq f\left(t,y(\alpha(t))\right) = \mathcal{A}y(t), \forall t \in \mathcal{R}_+$$

Similarly, $\mathcal{B}x(t) = \int_0^t (t-s)\, g\left(s,x(\mu(s))\right) ds$

$$\leq \int_0^t (t-s)\, g\left(s,y(\mu(s))\right) ds \leq \mathcal{B}y(t), \forall t \in \mathcal{R}_+$$

Implies that $\mathcal{A}$ and $\mathcal{B}$ are non-decreasing operators on $[p_1,p_2]$. Again definition (2.5.4) and hypothesis $(\mathfrak{B}7)$ implies that

$$p_1(t) \leq f\left(t,p_1(\alpha(t))\right) \int_0^t (t-s)\, g[s,p_1(\mu(s))] ds$$

$$\leq f\left(t,x(\alpha(t))\right) \int_0^t (t-s)\, g\left(s,x(\mu(s))\right) ds$$

$$\leq f\left(t,p_2(\alpha(t))\right) \int_0^t (t-s)\, g\left(s,p_2(\mu(s))\right) ds$$

$\leq p_2(t), \forall t \in \mathcal{R}_+$ and $x \in [p_1,p_2]$

As a result $p_1(t) \leq \mathcal{A}x(t)\mathcal{B}x(t) \leq p_2(t), \forall t \in \mathcal{R}_+$ and $x \in [p_1,p_2]$

Hence $\mathcal{A}x\mathcal{B}x \in [p_1,p_2], \forall x \in [p_1,p_2]$

Again $M = \|\mathcal{B}([p_1,p_2])\| = sup\{\|\mathcal{B}x\| : x \in [p_1,p_2]\}$

$$\leq sup\left\{sup_{t\in\mathcal{R}_+}\int_0^t \left|(t-s)\,g\left(s,x(\mu(s))\right)ds\right| : x \in [p_1,p_2]\right\}$$

$$\leq sup\left\{T\ sup_{t\in\mathcal{R}_+}\int_0^t \left|g\left(s,x(\mu(s))\right)ds\right| : x \in [p_1,p_2]\right\}$$

$\leq T\int_0^t k(s)\,ds \leq T\|k\|_{\mathcal{L}^1}$,Since $LM \leq LT\|k\|_{\mathcal{L}^1} \leq 1$

We apply theorem (2.5.1) to the operator equation $\mathcal{A}x\mathcal{B}x = x$ to yield that the SNFDE (2.1.1) has minimum and maximum positive solution on $\mathcal{R}_+$.
This completes the proof.

Conclusion: In this paper we have studied the existence and locally attractivity of solutions to the second order nonlinear quadratic functional differential equation in Banach Space by Hybrid Fixed Point Theory.

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