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LOCALLY ATTRACTIVE SOLUTION TO FRACTIONAL ORDER QUADRATIC FUNCTIONAL INTEGRAL EQUATION

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ABSTRACT

In this research paper, we discuss some results concerning the existence of solution and locally attractive solution for fractional order quadratic functional integral equation in R_+ *by using a hybrid fixed point theorem on Banch algebras due to B.C.Dhage.*

KEYWORDS: Banach algebra, Quadratic functional integral equation, Lipschitz constant,existence result, locally attractive solution.

1. INTRODUCTION:

Nonlinear quadratic integral equations appear very often, in many applications of real world problem. For examples, quadratic integral equations are often applicable in the theory of radioactive transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. The theory of integral equations of fractional order has recently received a lot of attention and constitutes a significant branch of nonlinear analysis. Numerous research papers and monographs devoted to differential and integral equations of fractional order have appeared (see [1-3, 5-7, 9-17]. These papers contain various types of existence results for equations of fractional order. In this paper, we study the existence of solution and locally attractive solution of the following nonlinear quadratic integral equation of fractional order.

Let $\alpha \in (0,1)$ and R denote the real numbers whereas R_+ be the set of nonnegative numbers. Consider the fractional order nonlinear quadratic functional integral equation (QFIE):

$$x(t) = g(t, x(\varphi_2(t)))\left[q(t) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{h(s, x(\varphi_3(s)))}{(t-s)^{1-\alpha}}\, ds \right] + f\left(t, x\left(\varphi_1(t)\right)\right) \qquad (1.1)$$

for all $t \in R_+$

where $q: R_+ \to R$, $h(t,x) = h: R_+ \times R \to R$ $f: R_+ \times R \to R$, $g(t,x) = g: R_+ \times R \to R$ and $\varphi_1, \varphi_2, \varphi_3: R_+ \to R_+$

By a solution of the QFIE (1.1) we mean a function $x \epsilon BC(R_+, R)$ that satisfies (1.1) on R_+ .

Where$x \epsilon BC(R_+, R)$ is the space of continuous and bounded real-valued functions defined on R_+ .

In this paper, we prove the locally attractive of the solution for QFIE (1) using a classical hybrid fixed point theorem of B.C.Dhage [2]. Following are some preliminary definitions and auxiliary results that will be used in the follows .

2. PRELIMINARIES:

Let $X = BC(R_+, R)$ be Banach algebras with norm $\|\cdot\|$ and let Ω be a subset of X. Let a mapping $A: X \to X$ be an operator and consider the following operator equation in X, namely,
$x(t) = (Ax)(t) \quad t \in R_+ \ (2.1)$.

Below we give different characterizations of the solutions for operator equation (2.1) on R_+.
We need the following definitions in the sequel.

Definition 2.1: We say that solutions of the equation (2.1) are locally attractive if there exists an Closed ball $B_r[0]$ in the space $BC(R_+, R)$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (2.1) belonging to $B_r[0] \cap \Omega$,we have that

$$\lim_{t \to \infty} (x(t) - y(t)) = 0 \qquad (2.2)$$

Definition 2.2: Let X be a Banach space. A mapping $A: X \to X$ is called Lipschitz if there is a constant $\alpha > 0$ such that $\|Ax - Ay\| \le \alpha \|x - y\|$ for all $x, y \in X$. If $\alpha < 1$, then A is called a contractionon X with the contraction constant α .

Definition 2.3: **(Dugundji and Granas [8])**. An operator A on a Banach space X into itself is called Compact if for any bounded subset S of X, A(S) is a relatively compact subset of X. If A is continuous and compact, then it is called completely continuous on X.

Let X be a Banach space with the norm $\|\cdot\|$ and Let $A: X \to X$ be an nonlinear operator Then A is called

(i) Compact if $A(X)$ is relatively compact subset of X;

(ii) totally bounded if $A(S)$ is a totally bounded subset of X for any bounded subset S of X

(iii) Completely continuous if it is continuous and totally bounded operator on X.

It is clear that every compact operator is totally bounded but the converse need not be true.

We seek the solutions of (1.1) in the space $BC(R_+, R)$ of continuous and bounded real-valued functions defined on R_+. Define a standard supremum norm $\|\cdot\|$ and a multiplication "." in $BC(R_+, R)$by, $\| x \| = \sup\{|x(t)|: t \epsilon R_+\}$(2.3)

$(xy)(t) = x(t)y(t),\ t \in R_+.(2.4)$

Clear, $BC(R_+, R)$becomes a Banach space with respect to the above norm and the multiplication in it. By $L^1(R_+, R)$ we denote the space of Lebesgue integrable functions on R_+ with the norm $\|.\|_{L^1}$ defined by $\|x\|_{L^1} = \int_0^\infty |x(t)| dt$ (2.5)

Denote by $L^1(a,b)$ be the space of Lebesgue integrable functions on the interval (a, b), which is equipped with the standard norm.

Definition 2.4: The Riemann-Liouville fractional integral of order β of the function $x(t)$ is defined by the formula: $I^\beta x(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{x(s)}{(t-s)^{1-\beta}} ds, t \in (a,b)$ (2.6)

Where $\Gamma(\beta)$ denote the gamma function.

It may be shown that the fractional integral operator I^β transforms the space $L^1(a,b)$ into itself and has some other properties (see [12-19])

Theorem 2.1: (Arzela-Ascoli theorem) If every uniformly bounded and equi-continuous sequence $\{f_n\}$ of functions in$C(J, R)$, then it has a convergent subsequence.

Theorem 2.2: A metric space X is compact iff every sequence in X has a convergent subsequence.

Now we apply following hybrid fixed point theorem of Dhage [2] for proving the existence result.

Theorem 2.3 (Dhage, [2]). Let S be a closed convex and bounded subset ofa Banach algebra X and let A,B,C: S → X be three operators such that

(a) A and C are Lipschitzian with Lipschitz constants α and β respectively,

(b) B is completely continuous, and

(c) A*x*B*x* + C*x*ϵ S for each xϵ S.

Then the operator equation A*x*B *x* + C*x*= x has a solution whenever α M+ β < 1,

Where $M = \|B(S)\| = \sup\{\|Bx\| : x \in S\}$

3. E XISTENCE RESULTS:

Definition 3.1. A mapping $h(t, x) = h: R_+ \times R \to R$ is said to be Caratheodory if

1. $t \to h(t,x)$ is measurable for all $x \in R$, and
2. $t \to h(t,x)$ is continuous almost everywhere for $t \in R_+$

Again a caratheodory function h is called L^1 – Caratheodory if

3. for each real number $r > 0$ there exists a function $h_r \in L^1(R_+, R)$ such that $|h(t,x)| \leq h_r(t)$ a.e. $t \in R_+$ for all $x \in R$ with $|x| \leq r$.

Finally, a Caratheodory function $h(t, x)$ is called L^1_R – Caratheodory if

4. there exist a function $h_1 \epsilon\, L^1(R_+, R)$ such that $|h(t,x)| \leq h_1(t)$ a.e. $t \in R_+$ for all $x \in R$

For convenience, the function h_1 is referred to as a bound function of h

We consider the following set of hypotheses in the sequel.

a) The function $\varphi_1, \varphi_2 : R_+ \to R_+$ are continuous.

b) The function $f(t,x) = f : R_+ \times R \to R$ is continuous and bounded with bound

$F = \sup_{(t,x)\in R_+\times R} |f(t,x)|$ there exists a bounded function n:$R_+ \to R_+$ with bound N Satisfying $|f(t,x) - f(t,y)| \leq n(t)|x-y|$ for all $t \in R_+$, and x,y$\in R$ and F$\to 0$ as t $\to \infty$.

c) The function $g(t,x) = g : R_+ \times R \to R$ is continuous and bounded with bound $G = Sup_{(t,\, x)\in R_+\times R} |g(t,x)|$ there exists a bounded function $l : R_+ \to R_+$ with bound L

Satisfying $|g(t,x) - g(t,y)| \leq l(t)|x-y|$ for all $t \in R_+$, and $x, y \in R$

d) $q : R_+ = [0, +\infty) \to R$ is continuous function on R_+; also $\lim_{t\to\infty} q(t) = 0$

e) The function $h(t,x) = h: R_+ \times R \to R$ satisfy caratheodory condition (i.e. meausrable in t for all $x \in R$ and continuous in x for all $t \in R_+$) and there exist a function $h_1 \epsilon\, L^1(R_+, R)$ such that $|h(t,x)| \leq h_1(t), (t,x) \in R_+ \times R$ $x \in R$ where $\lim\limits_{t\to\infty} \int_0^t h_1(s)ds = 0$.

In what follows we will assume additionally that the following conditions satisfied.

e) The uniform continuous function v:$R_+ \to R_+$ defined by the formulas$v(t) = \int_0^t \frac{h_1(s)}{(t-s)^{1-\alpha}} ds$ is bounded on R_+ and vanish at infinity that is $\lim_{t\to\infty} v(t) = 0$.

Remark 3.1: Note that if the hypothesis (d)and (e) hold, then there exist constants $K_1 > 0$ and $K_2 > 0$ such that: $K_1 = \sup\{q(t) : t \in R_+\}$, $K_2 = \sup \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h_1(s)}{(t-s)^{1-\alpha}} ds$

Theorem 3.1:Suppose that the hypotheses [a]-[f] holds. Furthermore if $L(K_1 + K_2) + N < 1, K_1$ and K_2 are defined remark 3.1, Then the QFIE (1.1) has a solution in the space $BC(R_+, R)$ .Moreover, solutions of the equation (1) are locally attractive on R_+.

Proof: By a solution of the QFIE (1) we mean a continuous function $x : R_+ \to R$ that satisfies QFIE (1) on R_+.

Let $X = BC(R_+, R)$ be Banach Algebras of all continuous and bounded real valued function on R_+ with the norm $\|x\| = \sup_{t\in R_+} |x(t)|$ (3.1)

We shall obtain the solution of QFIE (1) under some suitable conditions on the functions involved in (1).

Consider the closed ball $B_r[0]$ in X centered at origin 0 and of radius r, where

$$r = F + G\ [K_1 + K_2] > 0$$

Let us define three operators A and B *and* C on $B_r[0]$ by

$$Ax(t) = g(t, x(\varphi_2(t))) \quad (3.2)$$

$$Bx(t) = \left[q(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(s, x(\varphi_3(s)))}{(t-s)^{1-\alpha}} ds \right] \quad (3.3)$$

and

$$Cx(t) = f\left(t, x\left(\varphi_1(t)\right)\right) \quad (3.4)$$

for all $t \in R_+$

Since the hypotheses (c) holds, the mapping A is well defined and the function $A\,x$ is continuous and bounded on R_+ and (b) holds, the mapping C is well defined and the function $C\,x$ is continuous and bounded on R_+ Again the function q is continuous on R_+, the function $B\,x$ is also continuous and bounded in view of hypotheses(d)

Therefore*A*, *B* and *C*defines the operators $A, B, C : B_r[0] \to X$ .we shall show that *A* and *B*and *C*satisfy all the requirements of theorem 2.3 (Dhage'stheo. for three operators) on $B_r[0]$.

Step I: we show that *A* is Lipschitz on $B_r[0]$ .let $x, y \in B_r[0]$ be arbitrary, and then by hypothesis(c) , we get $|Ax(t) - Ay(t)| = | \ g(t, x(\varphi_2(t)) - \ g(t, y(\varphi_2(t))|$

$$\leq \boldsymbol{l(t)} | \ x(\varphi_2(t) - y(\varphi_2(t))|$$

$\leq L\|x - y\|$for all $t \in R_+$. (3.41)

Taking suprimum over t

$\|Ax - Ay\| \leq L\|x - y\|$ for all $x, y \in B_r[0]$ (3.5)

This shows that *A* is Lipschitz Operator on $B_r[0]$ with the Lipschitz constant *L*.

Next, we show that *C* is Lipschitz on $B_r[0]$ .let $x, y \in B_r[0]$ be arbitrary, and then by hypothesis (b), we get

$$|Cx(t) - Cy(t)| = | \ f(t, x(\varphi_1(t)) - \ f(t, y(\varphi_1(t))|$$

$$\leq \boldsymbol{n(t)} | \ x(\varphi_1(t)) - \ y(\varphi_1(t))|$$

$$\leq N \parallel x - y \parallel \text{for all } t \in R_+. \quad (3.6)$$

Taking suprimum over t

$\parallel Cx - Cy \parallel \leq N \parallel x - y \parallel$------(3.61)

for all $x, y \in B_r[0]$

This shows that *C* is Lipschitz Operator on $B_r[0]$ with the Lipschitz constant *N*

Step II: we show that *B* is completely continuous operator on $B_r[0]$.

Firstly we show that *B* is continuous on $B_r[0]$**.**

Case I: Suppose that $t \geq T$, there exist $T > 0$ andlet us fix arbitrary $\varepsilon > 0$ and take $x, y \in B_r[0]$ such that $\|x - y\| \leq \varepsilon$. Then

$$|(Bx)t - (By)t| \leq \left| q(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(s, x(\varphi_3(s)))}{(t-s)^{1-\alpha}} \, ds \ - q(t) - \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(s, y(\varphi_3(s)))}{(t-s)^{1-\alpha}} \, ds \ \right|$$

$$\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(s, x(\varphi_3(s)))}{(t-s)^{1-\alpha}}\, ds \; - \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(s, y(\varphi_3(s)))}{(t-s)^{1-\alpha}}\, ds \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \left[\int_0^t \frac{\left| h\left(s, x(\varphi_3(s))\right) \right|}{(t-s)^{1-\alpha}}\, ds \; + \int_0^t \frac{|h(s, y(\varphi_3(s)))|}{(t-s)^{1-\alpha}}\, ds \right]$$

$$\leq \frac{1}{\Gamma(\alpha)} \left[\int_0^t \frac{h_1\,(s)}{(t-s)^{1-\alpha}}\, ds \; + \int_0^t \frac{h_1\,(s)}{(t-s)^{1-\alpha}}\, ds \right]$$

$$\leq \frac{2}{\Gamma(\alpha)} \left[\int_0^t \frac{h_1\,(s)}{(t-s)^{1-\alpha}}\, ds \right]$$

$$\leq \frac{2\, v(t)}{\Gamma(\alpha)} \qquad (3.7)$$

for all $t \geq T$

There exists T > 0 s.t. $v(t) \leq \frac{\varepsilon\, \Gamma(\alpha)}{2}$.Since ε is an arbitrary, we derive that

$$\left| (Bx)(t) - (By)(t) \right| \leq \varepsilon \qquad (3.71)$$

CaseII: Further, let us assume that $t \in [0, T]$, then evaluating similarly to above we obtain the following estimate

$$|(Bx)t - (By)t| \leq \left| q(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(s, x(\varphi_3(s)))}{(t-s)^{1-\alpha}}\, ds \; - q(t) - \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(s, y(\varphi_3(s)))}{(t-s)^{1-\alpha}}\, ds \right|$$

$$\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(s, x(\varphi_3(s)))}{(t-s)^{1-\alpha}}\, ds \; - \frac{1}{\Gamma(\alpha)} \int_0^T \frac{h(s, y(\varphi_3(s)))}{(t-s)^{1-\alpha}}\, ds \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \left[\int_0^T \frac{\left| h\left(s, x(\varphi_3(s))\right) - h(s, y(\varphi_3(s))) \right|}{(T-s)^{1-\alpha}}\, ds \right]$$

$$\leq \frac{1}{\Gamma(\alpha)} \left[\int_0^T \frac{w_r^T\,(h, \epsilon)}{(T-s)^{\alpha}}\, ds \right]$$

$$\leq \frac{1}{\Gamma(\alpha)} \left[\frac{w_r^T\,(h, \epsilon)}{\alpha} T^{\alpha} ds \right]$$

$$\leq \left[\frac{w_r^T\ (h,\epsilon)}{\Gamma(\alpha+1)} T^{\alpha} ds\right] \qquad (3.8)$$

Where $w_r^T(h,\epsilon) = \sup\{|h(s,x) - h(s,y)| : s \in [0,T]; x,y \in [-r,r], |x,y| \leq \epsilon\}$

Therefore, from the uniform continuity of the function $h(t,x)$ on the set $[0,T]\times[-r,r]$ .we derive that $w_r^T(h,\epsilon) \to 0$ as $\varepsilon \to 0$.

Now from case I and II, we conclude that the operator B is continuous operator on closed ball $B_r[0]$ in to itself.

Step III: Next we show that B is compact on $B_r[0]$.

First prove that every sequence $\{Bx_n\}$ in $B(B_r[0])$ has a uniformly bounded sequence in $B(B_r[0])$. Now by(d)–(f)

$$|(Bx_n)t| = \left| q(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(s, x_n(\varphi_3(s)))}{(t-s)^{1-\alpha}}\ ds \right|$$

$$|(Bx_n)t| \leq |q(t)| + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{|h(s, x_n(\varphi_3(s)))|}{(t-s)^{1-\alpha}}\ ds$$

$$|(Bx_n)t| \leq |q(t)| + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h_1\ (s)}{(t-s)^{1-\alpha}}\ ds$$

$$|(Bx_n)t| \leq |q(t)| + \frac{v(t)}{\Gamma(\alpha)}$$

$$|(Bx_n)t| \leq K_1 + K_2 \text{ for all } t \in R_+ . \qquad (3.9)$$

Taking supremum over t, We obtain $\|Bx_n\| \leq K_1 + K_2$ for all $n \in N$.

This shows that $\{Bx_n\}$ is a uniformly bounded sequence in $B(B_r[0])$.

Now we proceed to show thatsequence $\{Bx_n\}$ is also equicontinuous.

Let $\varepsilon > 0$ be given. Since $\lim_{t\to\infty} q(t) = 0$, there is constant $T > 0$ such that $|q(t)| < \frac{\varepsilon}{2}$ for all $t \geq T$

i) let $t_1, t_2 \in R_+$ be arbitrary.If $t_1, t_2 \in [0,T]$, then we have

$$\left|Bx_n(t_2)-Bx_n(t_1)\right|$$

$$\leq \left| q(t_2) + \frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{h(s, x_n\ (\varphi_3(s)))}{(t_2 - s)^{1-\alpha}}\, ds\ \ - q(t_1) - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{h(s, x_n\ (\varphi_3(s)))}{(t_1 - s)^{1-\alpha}}\, ds \right|$$

$$\leq |q(t_2) - q(t_1)| + \left| \frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{|h(s,x_n\ (\varphi_3(s)))|}{(t_2 - s)^{1-\alpha}}\, ds\ \ - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{|h(s,x_n\ (\varphi_3(s)))|}{(t_1 - s)^{1-\alpha}}\, ds \right|$$

$$\leq |q(t_2) - q(t_1)| + \left| \frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{h_1(s)}{(t_2 - s)^{1-\alpha}}\, ds\ \ - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{h_1(s)}{(t_1 - s)^{1-\alpha}}\, ds \right|$$

$$\leq |q(t_2) - q(t_1)| + \frac{1}{\Gamma(\alpha)}\left| \int_0^{t_2} \frac{h_1(s)}{(t_2 - s)^{1-\alpha}}\, ds\ \ - \int_0^{t_1} \frac{h_1(s)}{(t_1 - s)^{1-\alpha}}\, ds \right|$$

$$\leq |q(t_2) - q(t_1)| + \frac{1}{\Gamma(\alpha)} |v(t_2) - v(t_1)| \qquad (3.10)$$

from the uniform continuity of the function $q(t)$ on $[0,T]$, we get $\left|Bx_n(t_2)-Bx_n(t_1)\right| \to 0$ as $t_1 \to t_2$.

ii) If $t_1, t_2 \geq T$ then we have

$$\left|Bx_n(t_2)-Bx_n(t_1)\right|$$

$$\leq \left| q(t_2) + \frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{h(s, x_n\ (\varphi_3(s)))}{(t_2 - s)^{1-\alpha}}\, ds\ \ - q(t_1) - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{h(s, x_n\ (\varphi_3(s)))}{(t_1 - s)^{1-\alpha}}\, ds \right|$$

$$\leq |q(t_2) - q(t_1)| + \left| \frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{h(s,x_n\ (\varphi_3(s)))}{(t_2 - s)^{1-\alpha}}\, ds\ \ - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{h(s,x_n\ (\varphi_3(s)))}{(t_1 - s)^{1-\alpha}}\, ds \right|$$

$$\leq |q(t_2) - q(t_1)| + \left| \frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{h(s,x_n\ (\varphi_3(s)))}{(t_2 - s)^{1-\alpha}}\, ds \right| + \left| \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{h(s,x_n\ (\varphi_3(s)))}{(t_1 - s)^{1-\alpha}}\, ds \right|$$

$$\leq |q(t_2) - q(t_1)| + \frac{v(t_2)}{\Gamma(\alpha)} + \frac{v(t_1)}{\Gamma(\alpha)} \leq 0 + \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$\leq \varepsilon \text{ as } t_1 \to t_2. \qquad (3.11)$$

iii) If t_1 , t_2 $\in$ R_+

With t_1 < T < t_2 then we have

$$|(Bx_n)t_2 - (Bx_n)t_1| \leq |Bx_n(t_2) - Bx_n(T)| + |Bx_n(T) - Bx_n(t_1)| \quad \textbf{(3.12)}$$

Now if $t_1 \to t_2$ $then$ $t_1 \to$T and T$\to t_2$

Therefore, $|Bx_n(t_2) - Bx_n(T)| \to 0$, $|Bx_n(T) - Bx_n(t_1)| \to 0$

and so $|(Bx_n)t_2 - (Bx_n)t_1| \to 0$ as $t_1 \to t_2$ for all $t_1, t_2 \in R_+$ **(3.13)**

Hence $\{Bx_n\}$ is an equicontinuous sequence of functions in X. Now an application of the Arzela-Ascoli theorem yields that $\{Bx_n\}$ has a uniformly convergent subsequence on the compact subset $[0,T]$ of R_+. Without loss of generality, call the subsequence of the sequence itself.This shows thatB is compact on $B_r[0]$.

Step IV: Next we show that $x = AxBx + Cx \in B_r[0] \Rightarrow x \in B_r[0]$ for all $x \in B_r[0]$

Let $x \in X$ be an arbitrary, such that $x = AxBx + Cx$

$$|A\,x(t)B\,x(t) + C\,x(t)| \leq |A\,x(t)||B\,x(t)| + |C\,x(t)|$$

$$\begin{aligned} |Ax(t)Bx(t) + Cx(t)| \\ &\leq |\ g(t, x(\varphi_2(t)))| \left\| \left[q(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h\left(s, x(\varphi_3(s))\right)}{(t-s)^{1-\alpha}}\, ds \right] \right\| \\ &+ \quad \left| f\left(t, x(\varphi_1(t))\right) \right| \end{aligned}$$

$$\leq G \left[|q(t)| + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{|h(s, x(\varphi_3(s)))|}{(t-s)^{1-\alpha}} ds \right] + F$$

$$\leq F + G \left[|q(t)| + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h_1(s)}{(t-s)^{1-\alpha}} ds \right]$$

$$\leq F + G \left[|q(t)| + \frac{v(t)}{\Gamma(\alpha)} \right]$$

$\leq F + G\ [K_1 + K_2] = r$ for all t in R₊ (3.14)

Taking the supremum over t, we obtain $\| AxBx + Cx \| \leq r$ for all $x \in B_r[0]$

Hence hypothesis (c) of Theorem holds.

Also we have $M = \left\| B\left(B_r[0]\right)\right\| = \sup\left\{\|Bx\| : x \in B_r[0]\right\}$

$$= sup\{ \underbrace{sup}_{t\geq 0} |q(t)| + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{|h(s,x(\varphi_3(s)))|}{(t-s)^{1-\alpha}} ds\} : x \in B_r[0] \}$$

$$= sup\{ \underbrace{sup}_{t\geq 0} |q(t)| + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h_1(s)}{(t-s)^{1-\alpha}} ds\} : x \in B_r[0]\}$$

$$\leq \underbrace{sup}_{t\geq 0} |q(t)| + \underbrace{sup}_{t\geq 0} \frac{v(t)}{\Gamma(\alpha)} \leq K_1 + K_2 \qquad (3.15)$$

Therefore M$\alpha + \beta$=L($K_1 + K_2$) + N< 1

Now Appling Theorem 2.3 to shows that QFIE (1.1) has a solution on R_+.

Step V: Finally, we show the local attractivity of the solutions for QFIE (1.1). Let x and y be any two solutions of the QFIE (1) in $B_r[0]$ defined on R_+. Then we have,

$$|x(t) - y(t)| \leq \left| g(t,x(\varphi_2(t)))\left[q(t) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{h(s,x(\varphi_3(s)))}{(t-s)^{1-\alpha}}\, ds \right] + f\left(t,x\left(\varphi_1(t)\right)\right) \right.$$
$$\left. - \; g(t,y(\varphi_2(t)))\left[q(t) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{h(s,y(\varphi_3(s)))}{(t-s)^{1-\alpha}}\, ds \right] - f\left(t,y\left(\varphi_1(t)\right)\right) \right|$$

$$|x(t) - y(t)| \leq | \; g(t,x(\varphi_2(t)))| \left\| \left[|q(t)| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{|h(s,x(\varphi_3(s)))|}{(t-s)^{1-\alpha}}\, ds \right] \right\|$$
$$+ \left| f\left(t,x\left(\varphi_1(t)\right)\right) \right|$$

$$+| \; g(t,y(\varphi_2(t)))| \left\| \left[|q(t)| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{|h(s,y(\varphi_3(s)))|}{(t-s)^{1-\alpha}}\, ds \right] \right\| + \left| f\left(t,y\left(\varphi_1(t)\right)\right) \right|$$

$$|x(t) - y(t)| \leq G \left\| \left[|q(t)| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{h_1(s)}{(t-s)^{1-\alpha}}\, ds \right] \right\| + F + G \left\| \left[|q(t)| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{h_1(s)}{(t-s)^{1-\alpha}}\, ds \right] \right\| + F$$

$$|x(t) - y(t)| \leq 2G \left\| \left[|q(t)| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{h_1(s)}{(t-s)^{1-\alpha}}\, ds \right] \right\| + 2F$$

$$|x(t) - y(t)| \leq 2G \left\| \left[|q(t)| + \frac{v(t)}{\Gamma(\alpha)} \right] \right\| + 2F$$

for all $t \in R_+$. Since $\lim_{t\to\infty} q(t) = 0$ and, $\lim_{t\to\infty} v(t) = 0$ and by hypothesis (b) , the above inequality it follows that $\lim_{t\to\infty} |x(t) - y(t)| = 0$. This completes the proof.

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