International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 3, Special Issue 1, July 2015, PP 328-336, ISSN 2347-307X (Print)
www.arcjournals.org

EXISTENCE OF LOCALLY ATTRACTIVE SOLUTION TO FRACTIONAL ORDERQUADRATIC FUNCTIONALINTEGRAL EQUATION

B.D. Karande, P.M.More & S.N.Kondekar
Department of Mathematics,Maharashtra UdayagiriMahavidyalaya, Maharashtra, India.

Abstract: In this article, we discuss the existence and uniqueness of locally attractive solution to fractional orderquadratic functional integral equation in R_+. Hybrid fixed point theorem are the main tool used here to establish the existence results

Keywords: Banach algebra, Quadratic functional integral equations, existence result, locally attractive solution.

1. INTRODUCTION

Nonlinear quadratic integral equations appear very often, in many applications of real world problem. For examples, quadratic integral equations are often applicable in the theory of radiactive transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. The theory of integral equations of fractional order has recently received a lot of attention and constitutes a significant branch of nonlinear analysis. Numerous research papers and monographs devoted to differential and integral equations of fractional order have appeared (see [1-3, 5-7, 9-17]. These papers contain various types of existence results for equations of fractional order. In this paper; we study the existence results of locally attractive solutions of the following nonlinear quadratic integral equation of fractional order.

Let $\beta \in (0,1)$ and R denote the real numbers whereas R_+ be the set of nonnegative numbers. Consider the nonlinear quadratic functional integral equation (QFIE) of fractional order:

$$x(t) = f\left(t, x(\alpha(t))\right)\left[q(t) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g\left(s, x(\gamma(s))\right)ds\right] \quad \forall t \in R_+ \quad (1.1)$$

where $q : R_+ \to R$, $f : R_+ \times R \to R$, $g(t,x) = g : R_+ \times R \to R$ and $\alpha, \gamma : R_+ \to R_+$.

By a solution of the FIE (1.1) we mean a function $x \in BC(R_+, R)$ that satisfies (1.1) on R_+ .

where $BC(R_+, R)$ is the space of continuous and bounded real-valued functions defined on R_+.

In this paper, we prove the locally attractive of the solution for QFIE (1.1) employing a classical hybrid fixed point theorem of Dhage [4]. In the next section, we collect some preliminary definitions and auxiliary results that will be used in the follows

2. PRELIMINARIES

Let $X = BC(R_+, R)$ be Banach algebra with norm $\|\cdot\|$ and let Ω be a subset of X. Let a mapping $A : X \to X$ be an operator and consider the following operator equation in X, namely,

$$x(t) = (Ax)(t) \qquad t \in R_+ \qquad (2.1)$$

Below we give different characterizations of the solutions for operator equation (2.1) on R_+.

We need the following definitions in the sequel.

Definition 2.1: We say that solutions of the equation (2.1) are locally attractive if there exists an $x_0 \in BC(R_+, R)$ and an $r > 0$ such that for all solutions $x = x(t)$ and $y = y(t)$ of equation (2.1) belonging to $B_r[x_0] \cap \Omega$ we have that $\lim_{t\to\infty}(x(t) - y(t)) = 0$ (2.2)

Definition 2.2: Let *X* be a Banach space. A mapping $A : X \to X$ is called Lipschitz if there is a constant $\alpha > 0$ such that $\|Ax - Ay\| \le \alpha \|x - y\|$ for all $x, y \in X$. If $\alpha < 1$, then *A* is called a **contraction** on *X* with the contraction constant α. **(2.3)**

Definition 2.3: (Dugundji and Granas [8]). An operator A on a Banach space X into itself is called Compact if for any bounded subset S of X, A(S) is a relatively compact subset of X. If A is continuous and compact, then it is called completely continuous on X.

Let *X* be a Banach space with the norm $\|.\|$ and Let $A : X \to X$ be an operator (in general nonlinear). Then *A* is called

(i) Compact if $A(X)$ is relatively comact subset of X;

(ii) totally bounded if $A(S)$ is a totally bounded subset of X for any bounded subset S of X

(iii) completely continuous if it is continuous and totally bounded operator on *X*.

It is clear that every compact operator is totally bounded but the converse need not be true.

We seek the solutions of (1.1) in the space $BC(R_+, R)$ of continuous and bounded real-valued functions defined on R_+. Define a standard supremum norm $\|.\|$ and a multiplication "." in $BC(R_+, R)$ by

$$\|x\| = \sup\left\{|x(t)| : t \in J\right\}, \quad (xy)(t) = x(t)y(t),\ t \in R_+. \tag{2.4}$$

Clear, $BC(R_+, R)$ becomes a Banach space with respect to the above norm and the multiplication in it. By $L^1(R_+, R)$ we denote the space of Lebesgueintegrable functions on R_+ with the norm $\|.\|_{L^1}$ defined by

$$\|x\|_{L^1} = \int_0^\infty |x(t)|dt \tag{2.5}$$

Denote by $L^1(a,b)$ be the space of Lebesgueintegrable functions on the interval (a, b), which is equipped with the standard norm. Let $x \in L^1(a,b)$ and let $\beta > 0$ be a fixed number.

Definition 2.4: The left sided Riemann-Liouville fractional integral [10, 12, 18] of order β of real function f is defined as

$$I_{a^+}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{f(t)}{(x-t)^{1-\beta}} dt \quad \beta > 0, x > a$$

Definition 2.5: The Riemann-Liouville fractional integral of order β of the function $x(t)$ is defined by the formula:

$$I^{\beta} x(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{x(s)}{(t-s)^{1-\beta}} ds, \ t \in (a,b) \tag{2.6}$$

Where $\Gamma(\beta)$ denote the gamma function.

It may be shown that the fractional integral operator I^{β} transforms the space $L^1(a,b)$ into itself and has some other properties (see [12-19])

Theorem 2.1: (Arzela-Ascoli theorem) If every uniformly bounded and equi-continuous sequence $\{f_n\}$ of functions in $C(J,R)$, then it has a convergent subsequence.

Theorem 2.2: A metric space X is compact iff every sequence in X has a convergent subsequence.

We employ a hybrid fixed point theorem of Dhage [4] for proving the existence result.

Theorem 2.3 :(Dhage [4]). Let S be a closed-convex and bounded subset of the Banach space X and let $A,B:S\to S$ be two operators satisfying:

(a) A is Lipschitz with the Lipschitz constant k,

(b) B is completely continuous,

(c) $AxBx\in S$ for all $x\in S$,and

(d) $Mk<1$ Where $M=\|B(S)\|=\sup\{\|Bx\|:x\in S\}$.

Then the operator equation $AxBx=x$ has a solution and the set of all solutions is compactin S

3. EXISTENCE RESULTS

Definition 3.1: A mapping $g:R_+\times R\to R$ is said to be Caratheodory if

1. $t\mapsto g(t,x)$ is measurable for all $x\in R$, and
2. $t\mapsto g(t,x)$ is continuous almost everywhere for $t\in R_+$

Again a caratheodory function *g* is called L^1-Caratheodory if

3. for each real number $r>0$ there exists a function $h_r\in L^1(R_+,R)$ such that $|g(t,x)|\le h_r(t)$ a.e. $t\in R_+$ for all $x\in R$ with $|x|\le r$.

Finally, a Caratheodory function $g(t,x)$ is called L^1_R-Caratheodory if

4. there exist a function $h\in L^1(R_+,R)$ such that $|g(t,x)|\le h(t)$ a.e. $t\in R_+$ for all $x\in R$

For convenience, the function *h* is referred to as a bound function of *g*
We consider the following set of hypotheses in the sequel.

(A_1) The function $\alpha:R_+\to R_+$ is continuous.

(A_2) The function $f(t,x)=f:R_+\times R\to R$ is continuous and bonded with bound $F=\sup_{(t,x)\in R_+\times R}|f(t,x)|$ there exists a bounded function $l:R_+\to R_+$ with bound L Satisfying $|f(t,x)-f(t,y)|\le l(t)|x-y|$ for all $t\in R_+$, and $x,y\in R$

(B_1) $q:R_+=[0,+\infty)\to R$ is continuous function on R_+; also $\lim_{t\to\infty}q(t)=0$

(B_2) The function $g(t,x)=g:R_+\times R\to R$ satisfy caratheodory condition (i.e.measrable in *t* for all $x\in R$ and continuous in x for all $t\in R_+$) and there exist function $h\in L^1$ Such that

$g(t,x)\le h(t)\forall(t,x)\in R_+\times R$ where $\lim_{t\to\infty}\int_0^t h(s)ds=0$

In what follows we will assume additionally that the following conditions satisfied.

(B_3) The function $a:R_+\to R_+$ defined by the formulas $a(t)=h(t)t^{\beta}$ is bounded on R_+ and vanish

at infinity, that is , $\lim_{t\to\infty}a(t)=0$, there is a real number $T>0$ such that $a(t)\le\frac{\Gamma(\beta+1)\varepsilon}{2}$ for all $t\ge T$

Remark 3.1: Note that if the hypothesis (B_1) and (B_2) hold, then there exist constants $K_1>0$ and $K_2>0$ such that: $K_1=\sup\{q(t):t\in R_+\}$, $K_2=\sup\left\{\frac{a(t)}{\Gamma(\beta+1)}:t\in R_+\right\}$

Theorem 3.1: Suppose that the hypotheses [(A_1)-(A_2)] and [(B_1)-(B_3)]are hold. Furthermore if $L(K_1+K_2)<1$, where K_1 and K_2 are defined remark 3.1, Then the QFIE (1.1) has a solution in the space $BC(R_+,R)$.Moreover, solutions of the equation (1.1) are locally attractive on R_+.

Proof: By a solution of the QFIE (1.1) we mean a continuous function $x:R_+\to R$ that satisfies QFIE (1.1) on R_+.

Let $X=BC(R_+,R)$ be Banach Algebra of all continuous and bounded real valued function on R_+ with the norm $\|x\|=\sup_{t\in R_+}|x(t)|$ $\quad(3.1)$

We shall obtain the solution of QFIE (1.1) under some suitable conditions on the functions involved in (1.1).

Consider the closed ball $B_r[0]$ in X centered at origin 0 and of radius r, where $r=F(K_1+K_2)>0$

Let us define two operators *A* and *B* on $B_r[0]$ by

$$Ax(t)=f(t,x(\alpha(t))) \tag{3.2}$$

and
$$Bx(t)=q(t)+\frac{1}{\Gamma(\beta)}\int_0^t\frac{g(s,x(\gamma(s)))}{(t-s)^{1-\beta}}ds \tag{3.3}$$

for all $t\in R_+$

Since the hypotheses (A_2) holds, the mapping *A* is well defined and the function *Ax* is continuous and bounded on R_+. Again the function *q* is continuous on R_+, the function *Bx* is also continuous and bounded in view of hypotheses (B_1).

Therefore *A* and *B* define the operators $A,B:B_r[0]\to X$ .we shall show that *A* and *B* satisfy all the requirements of theorem 2.3 on $B_r[0]$.

Step I: Firstly, we show that *A* is Lipschitz on $B_r[0]$.let $x,y\in B_r[0]$ be arbitrary, and then by hypothesis (A_2), we get $|Ax(t)-Ay(t)|=|f(t,x(\alpha(t)))-f(t,y(\alpha(t)))|$

$\le l(t)|x(\alpha(t))-y(\alpha(t))|$

$\le L\|x-y\|$ for all $t\in R_+$. $\quad(3.4)$

Taking supremum over t

$\|Ax-Ay\|\le L\|x-y\|$ for all $x,y\in B_r[0]$ $\quad(3.5)$

This shows that *A* is Lipschitz Operator on $B_r[0]$ with the Lipschitz constant *L*.

Step II: Secondly, we show that *B* is completely continuous operator on $B_r[0]$.

Firstly we show that *B* is continuous on $B_r[0]$.

CaseI: Suppose that $t\ge T$, there exist $T>0$ andlet us fix arbitrary $\varepsilon>0$ and take $x,y\in B_r[0]$ such that $\|x-y\|\le\varepsilon$. Then

$$\left|(Bx)(t)-(By)(t)\right| \le \left| q(t)+\frac{1}{\Gamma(\beta)}\int_0^t \frac{g\left(s,x\left(\gamma(s)\right)\right)}{(t-s)^{1-\beta}}ds - q(t)-\frac{1}{\Gamma(\beta)}\int_0^t \frac{g\left(s,y\left(\gamma(s)\right)\right)}{(t-s)^{1-\beta}}ds \right|$$

$$\le \left| \frac{1}{\Gamma(\beta)}\int_0^t \frac{g\left(s,x\left(\gamma(s)\right)\right)}{(t-s)^{1-\beta}}ds - \frac{1}{\Gamma(\beta)}\int_0^t \frac{g\left(s,y\left(\gamma(s)\right)\right)}{(t-s)^{1-\beta}}ds \right|$$

$$\le \frac{1}{\Gamma(\beta)}\int_0^t \frac{\left|g\left(s,y\left(\gamma(s)\right)\right)\right|}{(t-s)^{1-\beta}} + \frac{1}{\Gamma(\beta)}\int_0^t \frac{\left|g\left(s,y\left(\gamma(s)\right)\right)\right|}{(t-s)^{1-\beta}}ds$$

$$\le \frac{1}{\Gamma(\beta)}\int_0^t \frac{h(t)}{(t-s)^{1-\beta}}ds + \frac{1}{\Gamma(\beta)}\int_0^t \frac{h(t)}{(t-s)^{1-\beta}}ds$$

$$\le \frac{2}{\Gamma(\beta)}\int_0^t \frac{h(t)}{(t-s)^{1-\beta}}ds \le \frac{2h(t)}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}ds \le \frac{2h(t)}{\Gamma(\beta)}\left[\frac{(t-s)^{\beta}}{-\beta}\right]_0^t$$

$$\le \frac{2h(t)}{\Gamma(\beta)}\frac{t^{\beta}}{\beta} \le \frac{2h(t)t^{\beta}}{\Gamma(\beta+1)} \le \frac{2a(t)}{\Gamma(\beta+1)} \le \varepsilon \text{ for all } t \ge T \tag{3.6}$$

Since ε is an arbitrary, we derive that

$$\left|(Bx)(t)-(By)(t)\right| \le \varepsilon \tag{3.7}$$

CaseII: Further, let us assume that $t \in [0,T]$, then evaluating similarly to above we obtain the following estimate

$$\left|(Bx)(t)-(By)(t)\right| \le \left| q(t)+\frac{1}{\Gamma(\beta)}\int_0^t \frac{g\left(s,x\left(\gamma(s)\right)\right)}{(t-s)^{1-\beta}}ds - q(t)-\frac{1}{\Gamma(\beta)}\int_0^t \frac{g\left(s,y\left(\gamma(s)\right)\right)}{(t-s)^{1-\beta}}ds \right|$$

$$\le \left| \frac{1}{\Gamma(\beta)}\int_0^T \frac{g\left(s,x\left(\gamma(s)\right)\right)}{(t-s)^{1-\beta}}ds - \frac{1}{\Gamma(\beta)}\int_0^T \frac{g\left(s,y\left(\gamma(s)\right)\right)}{(t-s)^{1-\beta}}ds \right|$$

$$\le \frac{1}{\Gamma(\beta)}\int_0^T \frac{\left|g\left(s,x\left(\gamma(s)\right)\right)-g\left(s,y\left(\gamma(s)\right)\right)\right|}{(t-s)^{1-\beta}}ds$$

$$\le \frac{1}{\Gamma(\beta)}\int_0^T \frac{\omega_r^T(g,\varepsilon)}{(t-s)^{\beta}}ds \le \frac{\omega_r^T(g,\varepsilon)}{\Gamma(\beta)}\frac{T^{\beta}}{\beta} \le \frac{T^{\beta}\omega_r^T(g,\varepsilon)}{\Gamma(\beta+1)} \tag{3.8}$$

Where $\omega_r^T(g,\varepsilon)=\sup\left\{\left|g(s,x)-g(s,y)\right| : s\in[0,T]; x,y\in[-r,r], |x-y|\le\varepsilon\right\}$

Therefore, from the uniform continuity of the function $g(t,x)$ on the set $[0,T]\times[-r,r]$ .we derive that $\omega_r^T(g,\varepsilon)\to 0$ as $\varepsilon \to 0$.

Now combining the case I and II, we conclude that the operator B is continuous operator on closed ball $B_r[0]$ in to itself.

Step III: Next we show that B is compact on $B_r[0]$.

(A) First prove that every sequence $\{Bx_n\}$ in $B(B_r[0])$ has a uniformly bounded sequence in $B(B_r[0])$. Now by (B_1)-(B_3)

$$|Bx_n(t)| \le |q(t)| + \frac{1}{\Gamma(\beta)}\int_0^t \frac{|g(s,x_n(\gamma(s)))|}{(t-s)^{1-\beta}}ds$$

$$\le |q(t)| + \frac{1}{\Gamma(\beta)}\int_0^t \frac{h(t)}{(t-s)^{1-\beta}}ds \le |q(t)| + \frac{h(t)}{\Gamma(\beta)} \times \frac{t^\beta}{\beta}$$

$$\le |q(t)| + \frac{h(t)t^\beta}{\Gamma(\beta+1)} \le |q(t)| + \frac{a(t)}{\Gamma(\beta+1)} \le K_1 + K_2 \text{ for all } t \in R_+. \tag{3.9}$$

Taking supremum over t, We obtain $\|Bx_n\| \le K_1 + K_2$ for all $n \in N$.

This shows that $\{Bx_n\}$ is a uniformly bounded sequence in $B(B_r[0])$.

(B) Now we proceed to show thatsequence $\{Bx_n\}$ is also equicontinuous.

Let $\varepsilon > 0$ be given. Since $\lim_{t\to\infty} q(t) = 0$, there is constant $T > 0$ such that $|q(t)| < \frac{\varepsilon}{2}$ for all $t \ge T$

If $t_1, t_2 \in [0,T]$, then we have

$$|Bx_n(t_2) - Bx_n(t_1)|$$

$$\le \left| q(t_2) + \frac{1}{\Gamma(\beta)}\int_0^{t_2} \frac{g(s,x_n(\gamma(s)))}{(t_2-s)^{1-\beta}}ds - q(t_1) - \frac{1}{\Gamma(\beta)}\int_0^{t_1} \frac{g(s,x_n(\gamma(s)))}{(t_1-s)^{1-\beta}}ds \right|$$

$$\le |q(t_2) - q(t_1)| + \frac{1}{\Gamma(\beta)}\left| \int_0^{t_1} \frac{g(s,x_n(\gamma(s)))}{(t_2-s)^{1-\beta}}ds + \int_{t_1}^{t_2} \frac{g(s,x_n(\gamma(s)))}{(t_2-s)^{1-\beta}}ds - \int_0^{t_1} \frac{g(s,x_n(\gamma(s)))}{(t_1-s)^{1-\beta}}ds \right|$$

$$\le |q(t_2) - q(t_1)| + \frac{1}{\Gamma(\beta)}\int_0^{t_1} \frac{|g(s,x_n(\gamma(s)))|}{(t_2-s)^{1-\beta}}ds + \frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2} \frac{|g(s,x_n(\gamma(s)))|}{(t_2-s)^{1-\beta}}ds - \frac{1}{\Gamma(\beta)}\int_0^{t_1} \frac{|g(s,x_n(\gamma(s)))|}{(t_1-s)^{1-\beta}}ds$$

$$\le |q(t_2) - q(t_1)| + \frac{1}{\Gamma(\beta)}\int_0^{t_1} \frac{|g(s,x_n(\gamma(s)))|}{(t_2-s)^{1-\beta}}ds + \frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2} \frac{|g(s,x_n(\gamma(s)))|}{(t_2-s)^{1-\beta}}ds - \frac{1}{\Gamma(\beta)}\int_0^{t_1} \frac{|g(s,x_n(\gamma(s)))|}{(t_1-s)^{1-\beta}}ds$$

$$\le |q(t_2) - q(t_1)| + \frac{1}{\Gamma(\beta)}\int_0^{t_1} \frac{h(t_1)}{(t_2-s)^{1-\beta}}ds + \frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2} \frac{h(t_2)}{(t_2-s)^{1-\beta}}ds - \frac{1}{\Gamma(\beta)}\int_0^{t_1} \frac{h(t_1)}{(t_1-s)^{1-\beta}}ds$$

$$\le |q(t_2) - q(t_1)| + \frac{h(t_1)}{\Gamma(\beta)}\left[\frac{(t_2-s)^\beta}{-\beta}\right]_0^{t_1} + \frac{h(t_2)}{\Gamma(\beta)}\left[\frac{(t_2-s)\beta}{-\beta}\right]_{t_1}^{t_2} - \frac{h(t_1)}{\Gamma(\beta)}\left[\frac{(t_1-s)^\beta}{-\beta}\right]_0^{t_1}$$

$$\le |q(t_2) - q(t_1)| + \frac{h(t_1)}{\Gamma(\beta)}\left[\frac{(t_2-t_1)^\beta}{-\beta} + \frac{t_2^\beta}{\beta} - \frac{t_1^\beta}{\beta}\right] + \frac{h(t_2)}{\Gamma(\beta)}\left[\frac{(t_2-t_1)^\beta}{\beta}\right]$$

$$\le |q(t_2) - q(t_1)| + \frac{h(t_1)}{\Gamma(\beta+1)}\left[t_2^\beta - t_1^\beta - (t_2-t_1)^\beta\right] + \frac{h(t_2)}{\Gamma(\beta+1)}(t_2-t_1)^\beta \tag{3.10}$$

from the uniform continuity of the function $q(t)$ on $[0,T]$, we get $\left|Bx_n(t_2)-Bx_n(t_1)\right|\to 0$ as $t_1\to t_2$.

If $t_1,t_2\geq T$ then we have

$$\left|Bx_n(t_2)-Bx_n(t_1)\right|\leq\left|q(t_2)+\frac{1}{\Gamma(\beta)}\int_0^{t_2}\frac{g(s,x_n(s))}{(t_2-s)^{1-\beta}}ds-q(t_1)-\frac{1}{\Gamma(\beta)}\int_0^{t_1}\frac{g(s,x_n(\gamma(s)))}{(t_1-s)^{1-\beta}}ds\right|$$

$$\leq\left|q(t_2)-q(t_1)\right|+\frac{1}{\Gamma(\beta)}\left|\int_0^{t_2}\frac{g(s,x_n(s))}{(t_2-s)^{1-\beta}}ds-\int_0^{t_1}\frac{g(s,x_n(\gamma(s)))}{(t_1-s)^{1-\beta}}ds\right|\leq\varepsilon\text{ as } t_1\to t_2. \tag{3.11}$$

As a result, $\left|Bx_n(t_2)-Bx_n(t_1)\right|\to 0$ as $t_1\to t_2$. Hence $\{Bx_n\}$ is an equicontinuous sequence of functions in X. Now an application of the Arzela-Ascoli theorem yields that $\{Bx_n\}$ has a uniformly convergent subsequence on the compact subset $[0,T]$ of R_+. Without loss of generality, call the subsequence of the sequence itself.This shows thatB is compact on $B_r[0]$.

Step IV: Next we show that $AxBx\in B_r[0]$ for all $x\in B_r[0]$ is arbitrary, then

$$\left|Ax(t)Bx(t)\right|\leq\left|Ax(t)\right|\left|Bx(t)\right|$$

$$\leq\left|f(t,x(\alpha(t)))\right|\left(\left|q(t)+\frac{1}{\Gamma(\beta)}\int_0^t\frac{g(s,x(\gamma(s)))}{(t-s)^{1-\beta}}ds\right|\right)$$

$$\leq\left|f(t,x(\alpha(t)))\right|\left(\left|q(t)\right|+\frac{1}{\Gamma(\beta)}\int_0^t\frac{\left|g(s,x(\gamma(s)))\right|}{(t-s)^{1-\beta}}ds\right)$$

$$\leq F\left(\left|q(t)\right|+\frac{1}{\Gamma(\beta)}\int_0^t\frac{h(t)}{(t-s)^{1-\beta}}ds\right)\leq F\left(\left|q(t)\right|+\frac{h(t)}{\Gamma(\beta)}\frac{t^\beta}{\beta}\right)$$

$$\leq F\left(\left|q(t)\right|+\frac{h(t)t^\beta}{\Gamma(\beta+1)}\right)\leq F\left(\left|q(t)\right|+\frac{a(t)}{\Gamma(\beta+1)}\right)\leq F(K_1+K_2)=r\text{ for all } t\in R_+ \tag{3.12}$$

Taking the supremum over t, we obtain $\|AxBx\|\leq r$ for all $x\in B_r[0]$.

Hence hypothesis (c) of Theorem 2.3 holds.

Also we have $M=\left\|B(B_r[0])\right\|=\sup\{\|Bx\|:x\in B_r[0]\}$

$$=\sup\left\{\sup_{t\geq0}\left\{\left|q(t)\right|+\frac{1}{\Gamma(\beta)}\int_0^t\frac{\left|g(s,x(\gamma(s)))\right|}{(t-s)^{1-\beta}}ds\right\}:x\in B_r[0]\right\}$$

$$\leq\sup_{t\geq0}\left|q(t)\right|+\sup_{t\geq0}\left[\frac{h(t)t^\beta}{\Gamma(\beta+1)}\right]$$

$$\leq\sup_{t\geq0}\left|q(t)\right|+\sup_{t\geq0}\left[\frac{a(t)}{\Gamma(\beta+1)}\right]\leq K_1+K_2 \tag{3.13}$$

and therefore $Mk = L\left(K_1 + K_2\right) < 1$.

Now we apply Theorem 2.3 to conclude that QFIE (1.1) has a solution on R_+.

Step V: Finally, we show the local attractivity of the solutions for QFIE (1.1). Let x and y be any two solutions of the QFIE (1.1) in $B_r\left[0\right]$ defined on R_+. Then we have,

$$\left|x(t)-y(t)\right| \le \left| f\left(t,x\left(\alpha(t)\right)\right)\left(q(t)+\frac{1}{\Gamma(\beta)}\int_0^t \frac{g\left(s,x\left(\gamma(s)\right)\right)}{(t-s)^{1-\beta}}ds\right)\right|$$

$$+\left| f\left(t,y\left(\alpha(t)\right)\right)\left(q(t)+\frac{1}{\Gamma(\beta)}\int_0^t \frac{g\left(s,y\left(\gamma(s)\right)\right)}{(t-s)^{1-\beta}}ds\right)\right|$$

$$\le \left| f\left(t,x\left(\alpha(t)\right)\right)\right|\left(\left|q(t)\right|+\frac{1}{\Gamma(\beta)}\int_0^t \frac{\left|g\left(s,x\left(\gamma(s)\right)\right)\right|}{(t-s)^{1-\beta}}ds\right)$$

$$+\left| f\left(t,y\left(\alpha(t)\right)\right)\right|\left(\left|q(t)\right|+\frac{1}{\Gamma(\beta)}\int_0^t \frac{\left|g\left(s,y\left(\gamma(s)\right)\right)\right|}{(t-s)^{1-\beta}}ds\right)$$

$$\le F\left(\left|q(t)\right|+\frac{1}{\Gamma(\beta)}\int_0^t \frac{h(t)}{(t-s)^{1-\beta}}ds\right)+F\left(\left|q(t)\right|+\frac{1}{\Gamma(\beta)}\int_0^t \frac{h(t)}{(t-s)^{1-\beta}}ds\right)$$

$$\le 2F\left(\left|q(t)\right|+\frac{1}{\Gamma(\beta)}\int_0^t \frac{h(t)}{(t-s)^{1-\beta}}ds\right)\le 2F\left(\left|q(t)\right|+\frac{h(t)t^{\beta}}{\Gamma(\beta+1)}\right)\le 2F\left(\left|q(t)\right|+\frac{a(t)}{\Gamma(\beta+1)}\right) \tag{3.14}$$

For all $t \in R_+$. Since $\lim_{t\to\infty} q(t) = 0$ and $\lim_{t\to\infty} a(t) = 0$, for $\varepsilon > 0$, there are real numbers $T^{'} > 0$ and $T^{"} > 0$ such that $\left|q(t)\right| < \frac{\varepsilon}{4F}$ for all $t \ge T^{'}$ and $a(t) \le \frac{\Gamma(\beta+1)\varepsilon}{4F}$ for all $t \ge T^{"}$. If we choose $T^* = \max\left\{T^{'}, T^{"}\right\}$, then from the above inequality it follows that $\left|x(t)-y(t)\right| \le \varepsilon$ for all $t \ge T^*$. This completes the proof.

REFERENCES

1. A.A.Kilbas, HariM.Srivastava and Juan J.Trujillo, Theory and Applications Fractional Differential equations , North-Holland Mathematics Studies,204, Elsevier Science B.V., Amsterdam ,2006, MR2218073(2007a:34002).Zbl 1092.45003.
2. A.A.Kilbas, J.J.Trujillo, Differential equations of fractional order: Methods, results, Problems, I.Appl.Anal. Vol.78 (2001), pp.153-192.
3. A.Babakhani, V.Daftardar-Gejii, Existence of positive solutions of nonlinear fractional differential equations, J.Math.Appl. Vol.278 (2003), pp.434-442.
4. B.C,Dhage, A Fixed point theorem in Banach algebras involving three operators with applications, Kyungpook Math J. Vol.44 (2004),pp.145-155.

5. B.D.Karande, Existence of uniform global locally attractive solutions for fractional order nonlinear random integral equation, Journal of Global Research in Mathematical Archives , Vol.1, No.8, (2013) ,34-43.
6. B.D.Karande, Fractional Order Functional Integro-Differential Equation in Banach Algebras, Malaysian Journal of Mathematical Sciences, Volume 8(S),(2014), 1-16.
7. B.D.Karande, Global attractively of solutions for a nonlinear functional integral equation of fractional order in Banach Space, AIP Conf.Proc. “10th international Conference on Mathematical Problems in Engineering, Aerospace and Sciences”1637 (2014), 469-478.
8. Dugungi, A.Granas, Fixed point Theory, MonographieMath.,Warsaw, 1982.
9. H.M.Srivastava,R.K.Saxena, Operators of fractional integration and applications, Appl.Math.Comput. Vol.118 (2006), pp.147-156.
10. I.Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1993.
11. J.Banas,B.Rzepka, An application of measures of noncompactness in the study of asymptotic stability, Appl. Math. Lett. Vol.16 (2003), pp.1-6.
12. K.S.Miller, B.Ross, An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New Yark, 1993.
13. Lakshmikantham and A.S.Vatsala, Basic theory of fractional differential equations, Nonlinear Analysis, 69(2008), pp.2677-2682.
14. M.M.El-Borai and M.I.Abbas, on some integro-differential equations of fractional orders involving caratheodory nonlinearities, Int.J.Modern Math, Vol.2 (2007) No.1, pp. 41-52.
15. M.M.El-Borai and M.I.Abbas, Solvability of an infinite system of singular integral equations, SerdicaMath.J. Vol.33 (2007), pp.241-252.
16. M.M.El-Borai, W.G.El-Sayed and M.I.Abbas, Monotonic solutions of a class of quadratic singular integral equations of Voltra type, Int.J.Contemp.Math.Sci. Vol.2 (2007), pp.89-102.
17. Mohamed I. Abbas, on the existence of locally attractive solutions of a nonlinear quadratic voltra integral equation of fractional order, Advances in difference equations, Vol. (2010), pp. 1-11.
18. S.Samko, A.A.Kilbas, O.Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Amsterdam;1993
19. X.Hu,J.Yan, The global attractivity and asymptotic stability of solution of a nonlinear integral equation, J.Math.Anal. Appl.., 321(2006) pp.147-156.