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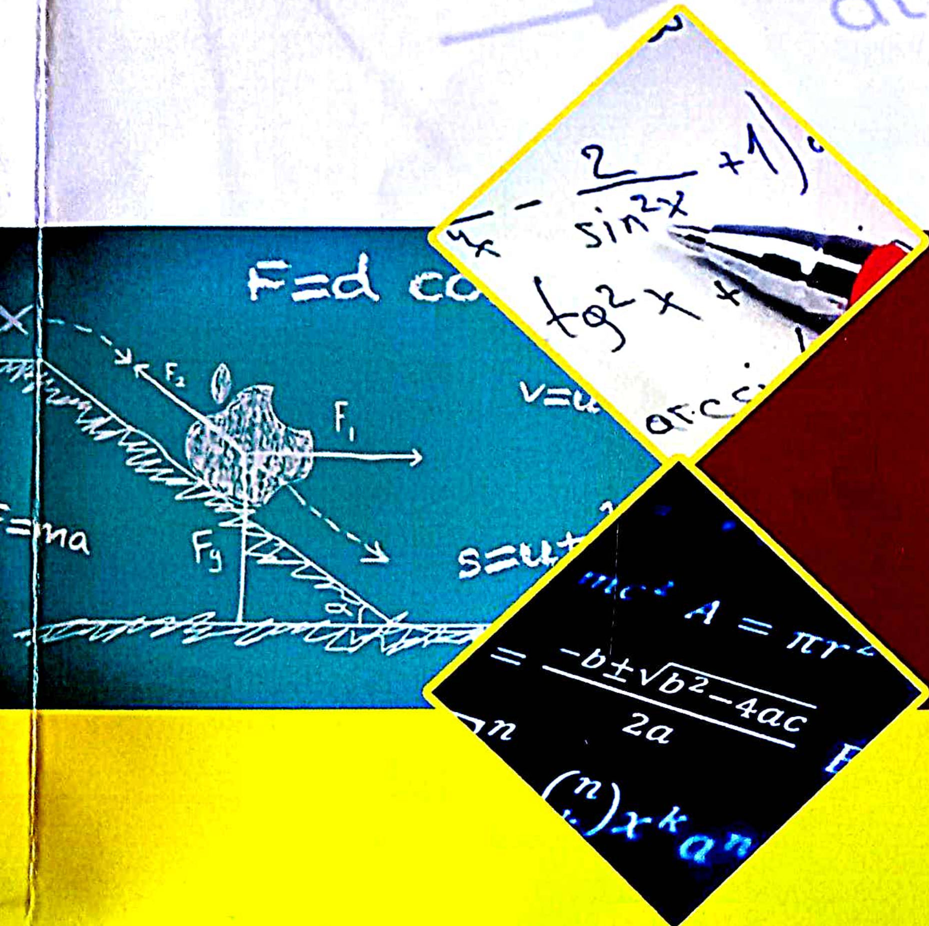

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EXISTENCE THE SOLUTION FOR FRACTIONAL ORDER QUADRATIC FUNCTIONAL INTEGRAL EQUATION

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ABSTRACT:

In this Paper, we discuss the existence the solution for fractional order quadratic functional integral equation in $\Re_+$. Krisonoselskii'sfixed point theorem is used here to establish the existence results.

Keywords: Banach algebras, Quadratic functional integral equations, existence result

INTRODUCTION:

Nonlinear functional-integral equations have been studied in the vehicular traffic, the biology, theory of optimal control and economics, etc. (Argyros, I.K., 1985). There are various cases of functional integral in literature, (Argyros, I.K., 1985; Deimling, K., 1985; Banas, J., B. Rzepka, 2003; XiaolingHu,Jurang Yan, 2006; Dhage, B.C., 2006; Maleknejad, K., 2008) and etc.

Nonlinear quadratic integral equations appear very often, in many applications of real world problem. For examples, quadratic integral equations are often applicable in the theory of radiactive transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. Numerous research papers and monographs devoted to differential and integral equations of fractional order have appeared (see [1, 2,3,17, 5-8, 13-18]. In the last 40 year or so, many authors have studied the existence of solutions for several classesof nonlinear quadratic integral equations. For example, Bana´s and Rzepka [21], Caballero *et al.* [18, 19], Darwish [25, 26, 27, 28, 29]These papers contain various types of existence results for equations of fractional order.

Here we study the existence results of of the following nonlinear quadratic integral equation of fractional order.

$$x(t) = \left[q(t) + \frac{1}{\Gamma(\tau)} \int_0^t \frac{g(s, x(\varphi_2(s)))}{(t-s)^{1-\tau}} ds\right] + f\left(t, x(\varphi_1(t))\right) \quad (*)$$

for all t $\in \Re_+$

Where $:\Re_+ \to \Re$, $f(t,x) = f: \Re_+ \times \Re \to \Re$, $g(t,x) = g: \Re_+ \times \Re \to \Re$ and $\varphi_1, \varphi_2 : \Re_+ \to \Re_+$

By a solution of the FIE (*) we mean a function $x \in BC(\Re_+, \Re)$that satisfies (*) on$\Re_+$. Where$BC(\Re_+, \Re)$ is the space of continuous and bounded real-valued functions defined on $\Re_+$ .In this paper, we prove the existence of the solution for QFIE (*) employingaKrisonoselkii's fixed point theorem. In the next section, we collect some preliminary definitions and auxiliary results that will be used in the follows.

PRELIMINARIES:

Let $X = BC(\mathfrak{R}_+, \mathfrak{R})$ be Banach algebra with norm $\|\cdot\|$ and let Ω be a subset of X. Let a mapping $\mathcal{A}: X \to X$ be an operator and consider the following operator equation in X, namely $x(t) = (\mathcal{A}x)(t)$ (2.1)

Below we give different characterizations of the solutions for operator equation (2.1) on $\mathfrak{R}_+$.

We need the following definitions in the sequel.

Definition 2.1[20]: Let X be a Banach space. A mapping $\mathcal{A} : X \to X$ is called Lipschitz if there is a constant $\alpha > 0$ such that $\|\mathcal{A}x - \mathcal{A}y\| \le \alpha\|x - y\|$ for all $x, y \in X$ If $\alpha < 1$ then $\mathcal{A}$ is called a **contraction** on X with the contraction constant α.

Definition 2.2: (Dugundji and Granas [13]). An operator $\mathcal{A}$ on a Banach space X into itself is called Compact if for any bounded subset S of X. $\mathcal{A}(S)$ is a relatively compact subset of X. If $\mathcal{A}$ is continuous and compact, then it is called completely continuous on X.

Let X be a Banach space with the norm $\|.\|$ and Let $\mathcal{A} : X \to X$ be an operator (in general nonlinear). Then $\mathcal{A}$ is called

(i) Compact if $\mathcal{A}(X)$ is relatively compact subset of X;

(ii) totally bounded if $\mathcal{A}(S)$ is a totally bounded subset of X for any bounded subset S of X

(iii) Completely continuous if it is continuous and totally bounded operator on X.

It is clear that every compact operator is totally bounded but the converse need not be true.

The solutions of (*) in the space $BC(\mathfrak{R}_+, \mathfrak{R})$ of continuous and bounded real-valued functions defined on R_+. Define a standard supremum norm $\|.\|$ and a multiplication "." in $BC(\mathfrak{R}_+, \mathfrak{R})$ by $\|x\| = \sup\{|x(t)|: t \in \mathfrak{R}_+\}$ (2.2)

$(xy)(t) = x(t)y(t) \quad t \in \mathfrak{R}_+$ (2.3)

Definition 2.4[17]: The Riemann-Liouville fractional integral of order β of the function $x(t)$ is defined by the formula: $I^\beta x(t) = \frac{1}{\Gamma(\beta)}\int_0^t \frac{x(s)}{(t-s)^{1-\beta}} ds \quad t \in (a, b)$ (2.5)

Where $\Gamma(\beta)$ denote the gamma function.

It may be shown that the fractional integral operator I^β transforms the space $\mathcal{L}^1(a, b)$ into itself and has some other properties (see [12-19])

Definition 2.5[17]: The left sided Riemann-Liouville fractional integral [10, 12, 18] of order β of real function f is defined as

$$I_{a^+}^{\beta} f(x) = \frac{1}{\Gamma(\beta)}\int_a^x \frac{f(t)}{(t-s)^{1-\beta}} dt \quad \beta > 0, x > a \qquad (2.4)$$

Theorem 2.1: (Arzela-Ascoli theorem) (8):If every uniformly bounded and equi-continuous sequence $\{ f_n \}$ of functions in $C(\mathcal{J}, \mathfrak{R})$, then it has a convergent subsequence.

Theorem 2.2[8]: A metric space X is compact iff every sequence in X has a convergent subsequence.

We apply a Krasnoselskii's fixed point theorem for proving the existence result.

EXISTENCE RESULTS:

Definition 3.1:. A mapping $g: \mathfrak{R}_+ \times \mathfrak{R} \to \mathfrak{R}$ is said to be Caratheodory if

1. $t \to g(t, x)$ is measurable for all $x \in \mathfrak{R}$, and
2. $x \to g(t, x)$ is continuous almost everywhere for $t \in \mathfrak{R}_+$

Again a caratheodory function g is called $\mathcal{L}^1$-Caratheodory if

3. for each real number $r > 0$ there exists a function $h_r \in \mathcal{L}^1(\mathfrak{R}_+, \mathfrak{R})$ such that $|g(t,x)| \le h_r(t)$ a.e. $t \in \mathfrak{R}_+$ for all $x \in \mathfrak{R}$ with $|x| \le r$

Finally, a Caratheodory function $g(t,x)$ is called $\mathcal{L}^1_{\mathfrak{R}}$ – Caratheodory if

4. there exist a function $h \in \mathcal{L}^1(\mathfrak{R}_+, \mathfrak{R})$ such that $|g(t,x)| \le h(t)$ a.e. $t \in \mathfrak{R}_+$ for all $x \in \mathfrak{R}$

For convenience, the function h is referred to as a bound function of g.

Theorem 3.1 :(Krisonoselskii's)(31,32,40) *Let X be a Banach Space and D be a bounded closed convex subset of X. Let $\mathcal{A}$, $\mathcal{B}$ maps D into X s.t. $\mathcal{A}u + \mathcal{B}u \in D$ for every (u, v) $\in$ D. If $\mathcal{A}$ is a contraction and $\mathcal{B}$ is completely continuous then the equation $\mathcal{A}w + \mathcal{B}w = w$ has a solution w on D. i.e.*

a) $\mathcal{A}$ is a contraction

b) $\mathcal{B}$ is completely continuous

c) $\mathcal{A}u + Bu \in D$.

Hypothesis: We consider the following set of hypotheses

a) The function $\varphi_1 : \mathfrak{R}_+ \to \mathfrak{R}_+$ is continuous.

b) $q: \mathfrak{R}_+ \to \mathfrak{R}$ is continuous function on $\mathfrak{R}_+$; also $\lim_{t\to\infty} q(t) = 0$

c) The function $f: \mathfrak{R}_+ \times \mathfrak{R} \to \mathfrak{R}$ is continuous and bounded with bound $F = \underset{(t,x)\in\mathfrak{R}_+\times\mathfrak{R}}{sup} |f(t,x)|$ there exists a bounded function $l : \mathfrak{R}_+ \to \mathfrak{R}_+$ with bound L with $\|L\| <$ 1 Satisfying $|f(t,x) - f(t,y)| \le l(t)|x-y|$ $t \in \mathfrak{R}_+$ for all $x, y \in \mathfrak{R}$

d) The function $g: \mathfrak{R}_+ \times \mathfrak{R} \to \mathfrak{R}$ satisfy caratheodory condition (i.e.measrable in t for all $x \in \mathfrak{R}$ and continuous in x for all $t \in \mathfrak{R}_+$) and there exist function $h \in \mathcal{L}^1(\mathfrak{R}_+, \mathfrak{R})$ Such that $g(t,x) \le h(t) \forall (t,x) \in \mathfrak{R}_+ \times \mathfrak{R}$ where $t \in \mathfrak{R}_+$ for all $x \in \mathfrak{R}$ where $\lim_{t\to\infty} \int_0^t h(s)ds = 0$

additionally we will assume following conditions are satisfied.

e) The uniform continuous function $v: \mathfrak{R}_+ \to \mathfrak{R}_+$ defined by the formulas $v(t) = \int_0^t \frac{h(s)}{(t-s)^{1-\tau}} ds$ is bounded on $\mathfrak{R}_+$ and vanish at infinity, that is, $\lim_{t\to\infty} v(t) = 0$.

Remark 3.1: Note that if the hypothesis (b) and (d) hold, then there exist constants $K_1 > 0$ and $K_2 > 0$ such that: : $K_1 = sup\{q(t): t \in \mathfrak{R}_+\}$. $K_2 = \sup \frac{1}{\Gamma(\tau)} \int_0^t \frac{h(s)}{(t-s)^{1-\tau}} ds$

Theorem 3.2: Suppose that the Hypothesis [(a) – (e)] holds .Furthermore if $L(K_1 + K_2) < 1$ where K_1 and K_2 are defined in remark 3.1 . Then the QFIE (*) has a solution in the Space $BC(\mathfrak{R}_+, \mathfrak{R})$.

Proof: Consider the Given Problem

$$x(t) = \left[q(t) + \frac{1}{\Gamma(\tau)} \int_0^t \frac{g(s, x(\varphi_2(s)))}{(t-s)^{1-\tau}} ds\right] + f\left(t, x(\varphi_1(t))\right)$$

for all $t \in \mathfrak{R}_+$

Where $q: \mathfrak{R}_+ \to \mathfrak{R}$, $f(t,x) = f: \mathfrak{R}_+ \times \mathfrak{R} \to \mathfrak{R}$, $g(t,x) = g: \mathfrak{R}_+ \times \mathfrak{R} \to \mathfrak{R}$ and $\varphi_1, \varphi_2 : \mathfrak{R}_+ \to \mathfrak{R}_+$

Now we define two operators $\mathcal{A}, \mathcal{B} : D \to X$ s.t.

$$\mathcal{A}x(t) = f\left(t, x(\varphi_1(t))\right)$$

$$\mathcal{B}x(t) = \left[q(t) + \frac{1}{\Gamma(\tau)} \int_0^t \frac{g(s, x(\varphi_2(s)))}{(t-s)^{1-\tau}} ds\right]$$

Consider the closed ball $B_r[0]$ in X centered at origin 0 and of radius r, where

$$r = F + (K_1 + K_2) > 0$$

We show that $\mathcal{A}$ and $\mathcal{B}$ satisfy all the requirements of theorem 3.1 on D

I]$\mathcal{A}$ is Contraction

Let $x, y \in X$ be arbitrary, and then by hypothesis(b) and (c), we get

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = |f(t, x(\varphi_1(t)) - f(t, y(\varphi_1(t))|$$
$$\leq l(t)|x(\varphi_1(t) - y(\varphi_1(t))|$$

$\leq L\|x - y\|$ for all $t \in \Re_+$

Taking supremum over t

$\|\mathcal{A}x - \mathcal{A}y\| \leq L\|x - y\|$ for all $x, y \in X$

This shows that $\mathcal{A}$ is contraction mapping with thecontraction constant L.

II] To show that $\mathcal{B}$ is continuous and compact operator

Firstwe show that $\mathcal{B}$ is continuous on D

CaseI: Suppose that $t \geq T$ there exist $T > 0$ and let us fix arbitrary$\varepsilon > 0$ and take $x, y \in$ Dsuch that$\|x - y\| \leq \varepsilon$. Then

$$|(\mathcal{B}x)t - (\mathcal{B}y)t| \leq \left|q(t) + \frac{1}{\Gamma(\tau)}\int_0^t \frac{g(s, x(\varphi_2(s)))}{(t-s)^{1-\tau}}ds - q(t) - \frac{1}{\Gamma(\zeta)}\int_0^t \frac{g(s, y(\varphi_2(s)))}{(t-s)^{1-\tau}}ds\right|$$

$$\leq \left|\frac{1}{\Gamma(\tau)}\int_0^t \frac{g(s, x(\varphi_2(s)))}{(t-s)^{1-\tau}}ds - \frac{1}{\Gamma(\tau)}\int_0^t \frac{g(s, y(\varphi_2(s)))}{(t-s)^{1-\tau}}ds\right|$$

$$\leq \frac{1}{\Gamma(\tau)}\left[\int_0^t \frac{\left|h\left(s, x(\varphi_2(s))\right)\right|}{(t-s)^{1-\tau}}ds + \int_0^t \frac{|h(s, y(\varphi_2(s)))|}{(t-s)^{1-\tau}}ds\right]$$

$$\leq \frac{1}{\Gamma(\tau)}\left[\int_0^t \frac{h(s)}{(t-s)^{1-\tau}}ds + \int_0^t \frac{h(s)}{(t-s)^{1-\tau}}ds\right]$$

$$\leq \frac{2}{\Gamma(\tau)}\left[\int_0^t \frac{h(s)}{(t-s)^{1-\tau}}ds\right]$$

$$\leq \frac{2\,v(t)}{\Gamma(\tau)}$$

for all $t \geq T$

There exists T > 0 s.t.$v(t) \leq \frac{\varepsilon\Gamma(\tau)}{2}$ Since $\varepsilon$is an arbitrary, we derive that

$$|(\mathcal{B}x)t - (\mathcal{B}y)t| \leq \varepsilon$$

CaseII: Further, let us assume that$t \in [0, T]$, then evaluating similarly to above we obtain the following estimate

$$|(\mathcal{B}x)t - (\mathcal{B}y)t| \leq \left|q(t) + \frac{1}{\Gamma(\tau)}\int_0^t \frac{g(s, x(\varphi_2(s)))}{(t-s)^{1-\tau}}ds - q(t) - \frac{1}{\Gamma(\tau)}\int_0^t \frac{g(s, y(\varphi_2(s)))}{(t-s)^{1-\tau}}ds\right|$$

$$\leq \left|\frac{1}{\Gamma(\tau)}\int_0^T \frac{g(s, x(\varphi_2(s)))}{(t-s)^{1-\tau}}ds - \frac{1}{\Gamma(\tau)}\int_0^T \frac{g(s, y(\varphi_2(s)))}{(t-s)^{1-\tau}}ds\right|$$

$$\leq \frac{1}{\Gamma(\tau)}\left[\int_0^T \frac{\left|g\left(s, x(\varphi_2(s))\right) - g(s, y(\varphi_2(s)))\right|}{(t-s)^{1-\tau}}ds\right]$$

$$\leq \frac{1}{\Gamma(\tau)}\left[\int_0^T \frac{w_r^T(g, \epsilon)}{(t-s)^{\tau}}ds\right]$$

$$\leq \frac{1}{\Gamma(\tau)}\left[\frac{w_r^T(g, \epsilon)}{\tau}T^{\tau}ds\right]$$

$$\leq \left[\frac{w_r^T(g,\epsilon)}{\Gamma(\tau+1)} T^\tau ds\right]$$

Where

$w_r^T(g,\epsilon) = \sup\{|g(s,x) - g(s,y)| : s \in [0,T]; x,y \in [-r,r], |x-y| \leq \epsilon\}$

Therefore, from the uniform continuity of the function $g(t,x)$ on the set $[0,T]X[-r,r]$ .we derive that $w_r^T(g,\epsilon) \to 0$ as $\varepsilon \to 0$.

Now from case I and II, we conclude that the operator $\mathcal{B}$ is continuous operator on closed ball D in to itself.

Step III: Next we show that $\mathcal{B}$ is compact on D

(A) First prove that every sequence $\{\mathcal{B}x_n\}$ inD has a uniformly bounded sequence inD. Now by(b) –(e)

$$|(\mathcal{B}x_n)t| = \left|q(t) + \frac{1}{\Gamma(\tau)}\int_0^t \frac{g(s,x_n(\varphi_2(s)))}{(t-s)^{1-\tau}} ds\right|$$

$$|(\mathcal{B}x_n)t| \leq |q(t)| + \frac{1}{\Gamma(\tau)}\int_0^t \frac{|g(s,x_n(\varphi_2(s)))|}{(t-s)^{1-\tau}} ds$$

$$|(\mathcal{B}x_n)t| \leq |q(t)| + \frac{1}{\Gamma(\tau)}\int_0^t \frac{h(s)}{(t-s)^{1-\tau}} ds$$

$$|(\mathcal{B}x_n)t| \leq |q(t)| + \frac{v(t)}{\Gamma(\tau)}$$

$$|(\mathcal{B}x_n)t| \leq K_1 + K_2 \forall t \in \Re_+$$

Taking supremum over t, we obtain $\|\mathcal{B}x_n\| \leq K_1 + K_2 \forall n \in N$

This shows that $\{\mathcal{B}x_n\}$ is a uniformly bounded sequence in D

(B) Now to that show the sequence $\{\mathcal{B}x_n\}$is equicontinuous.

Let $\varepsilon > 0$ b be given. Since $\lim_{t\to\infty} q(t) = 0$ there is constant $T > 0$ such that $|q(t)| < \varepsilon/2$ for all $t \geq T$

i) If t_1 , $t_2 \in [0,T]$ then we have

$|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1|$

$$\leq \left|q(t_2) + \frac{1}{\Gamma(\tau)}\int_0^{t_2} \frac{g(s,x_n(\varphi_2(s)))}{(t_2-s)^{1-\tau}} ds - q(t_1) - \frac{1}{\Gamma(\tau)}\int_0^{t_1} \frac{g(s,x_n(\varphi_2(s)))}{(t_1-s)^{1-\tau}} ds\right|$$

$$\leq |q(t_2) - q(t_1)| + \left|\frac{1}{\Gamma(\tau)}\int_0^{t_2} \frac{|g(s,x_n(\varphi_2(s)))|}{(t_2-s)^{1-\tau}} ds - \frac{1}{\Gamma(\tau)}\int_0^{t_1} \frac{|g(s,x_n(\varphi_2(s)))|}{(t_1-s)^{1-\tau}} ds\right|$$

$$\leq |q(t_2) - q(t_1)| + \left|\frac{1}{\Gamma(\tau)}\int_0^{t_2} \frac{h(s)}{(t_2-s)^{1-\tau}} ds - \frac{1}{\Gamma(\tau)}\int_0^{t_1} \frac{h(s)}{(t_1-s)^{1-\tau}} ds\right|$$

$$\leq |q(t_2) - q(t_1)| + \frac{1}{\Gamma(\tau)}\left|\int_0^{t_2} \frac{h(s)}{(t_2-s)^{1-\tau}} ds - \int_0^{t_1} \frac{h(s)}{(t_1-s)^{1-\tau}} ds\right|$$

$$\leq |q(t_2) - q(t_1)| + \frac{1}{\Gamma(\tau)}|v(t_2) - v(t_1)|$$

by the uniform continuity of the function $q(t), v(t)$ on $[0,T]$, we get $|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1| \to 0$ as $t_1 \to t_2$

ii) If t_1 , $t_2 \geq T$ then we have

$|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1|$

$$\leq \left|q(t_2) + \frac{1}{\Gamma(\tau)}\int_0^{t_2} \frac{g(s,x_n(\varphi_2(s)))}{(t_2-s)^{1-\tau}} ds - q(t_1) - \frac{1}{\Gamma(\tau)}\int_0^{t_1} \frac{g(s,x_n(\varphi_2(s)))}{(t_1-s)^{1-\tau}} ds\right|$$

$$\leq |q(t_2)-q(t_1)| + \left| \frac{1}{\Gamma(\tau)} \int_0^{t_2} \frac{g(s,x_n(\varphi_2(s)))}{(t_2-s)^{1-\tau}} ds - \frac{1}{\Gamma(\tau)} \int_0^{t_1} \frac{g(s,x_n(\varphi_2(s)))}{(t_1-s)^{1-\tau}} ds \right|$$

$$\leq |q(t_2)-q(t_1)| + \left| \frac{1}{\Gamma(\tau)} \int_0^{t_2} \frac{g(s,x_n(\varphi_2(s)))}{(t_2-s)^{1-\tau}} ds \right| + \left| \frac{1}{\Gamma(\tau)} \int_0^{t_1} \frac{g(s,x_n(\varphi_2(s)))}{(t_1-s)^{1-\tau}} ds \right|$$

$$\leq |q(t_2)-q(t_1)| + \frac{v(t_2)}{\Gamma(\tau)} + \frac{v(t_1)}{\Gamma(\tau)} \leq 0 + \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$\leq \epsilon$ as $t_1 \to t_2$.

iii) If $t_1, t_2 \in \Re_+$

With $t_1 < T < t_2$ then we have

$$|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1| \leq |\mathcal{B}x_n(t_2) - \mathcal{B}x_n(T)| + |\mathcal{B}x_n(T) - \mathcal{B}x_n(t_1)|$$

Now if $t_1 \to t_2$ then $t_1 \to T$ and $T \to t_2$

Therefore, $|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(T)| \to 0$, $|\mathcal{B}x_n(T) - \mathcal{B}x_n(t_1)| \to 0$

And so $|(\mathcal{B}x_n)t_2 - (\mathcal{B}x_n)t_1| \to 0$ as $t_1 \to t_2$ for all $t_1, t_2 \in \Re_+$

Hence $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in X. So by the Arzela-Ascoli theorem $\{\mathcal{B}x_n\}$has a uniformly convergent subsequence on the compact subset [0,T]of $\Re_+$. We call the subsequence of the sequence itself. This Yields that $\mathcal{B}$ is compact on D

So $\mathcal{B}$is completely continuous.

Step IV: Next we show that $\mathcal{A}x + \mathcal{B}x \in D$

$x \in D$ is arbitrary, then

$|\mathcal{A}x(t) + \mathcal{B}x(t)| \leq |\mathcal{A}x(t)| + |\mathcal{B}x(t)|$

$$|\mathcal{A}x(t) + \mathcal{B}x(t)| \leq \left\| \left[q(t) + \frac{1}{\Gamma(\tau)} \int_0^t \frac{g(s,x(\varphi_2(s)))}{(t-s)^{1-\tau}} ds \right] \right\| + \left| f\left(t, x(\varphi_1(t))\right) \right|$$

$$\leq \left[|q(t)| + \frac{1}{\Gamma(\zeta)} \int_0^t \frac{|g(s,x(\varphi_2(s)))|}{(t-s)^{1-\tau}} ds \right] + F$$

$$\leq F + \left[|q(t)| + \frac{1}{\Gamma(\zeta)} \int_0^t \frac{h(s)}{(t-s)^{1-\tau}} ds \right]$$

$$\leq F + \left[|q(t)| + \frac{v(t)}{\Gamma(\tau)} \right]$$

$\leq F + [K_1 + K_2] = r$ for all t in $\Re_+$

Taking the supremum over t, we obtain $\| \mathcal{A}x + \mathcal{B}x \| \leq r$ for all $x \in D$

Hence hypothesis (c) of Theorem holds.Now Appling Krisnoselskii'sTheorem [3.1] gives that QFIE (*) has a solution on $\Re_+$.

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