Basics of Calculus

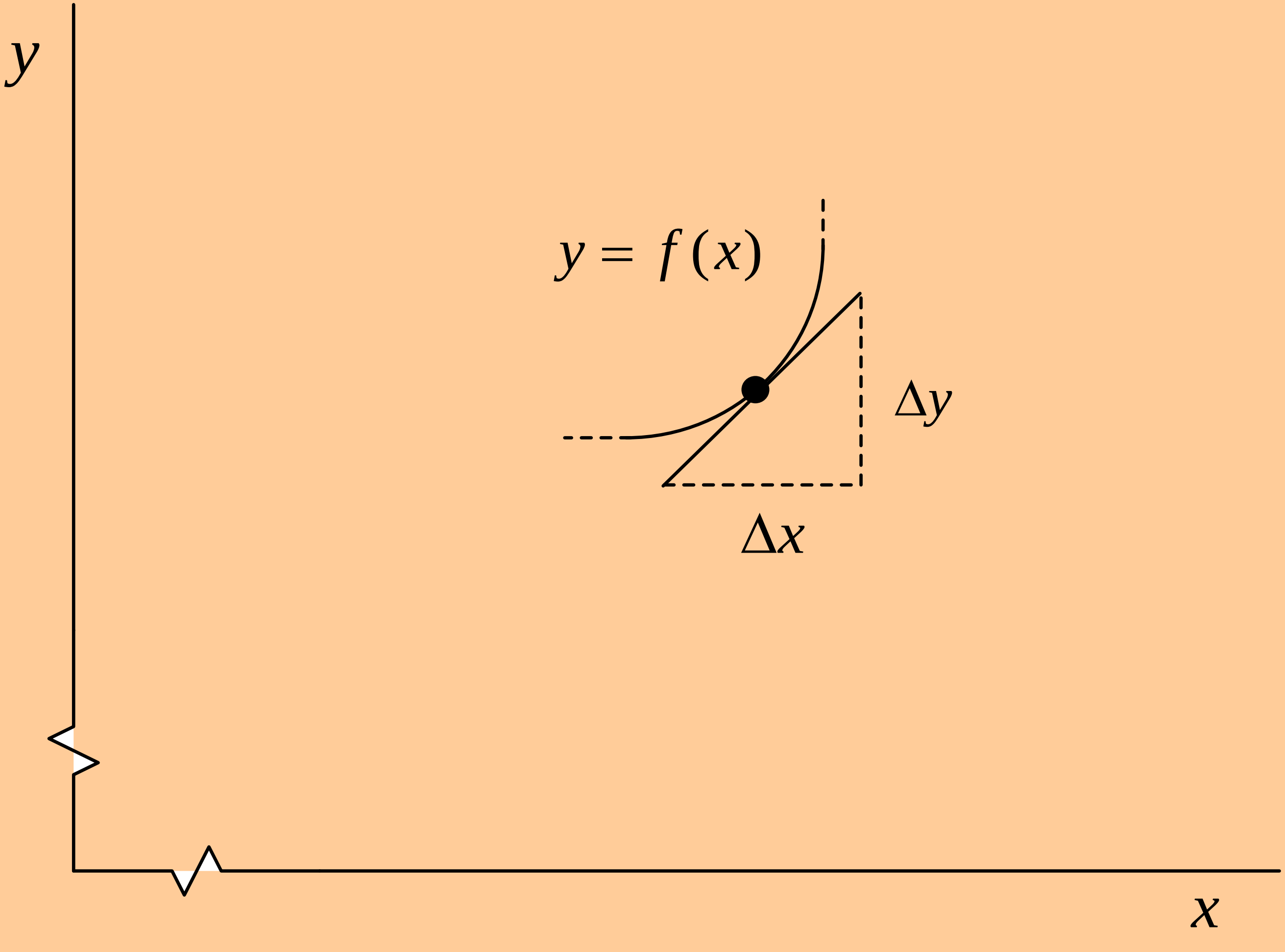
Differential Calculus

The two basic forms of calculus are *differential calculus* and *integral calculus*. This chapter will be devoted to the differential calculus.

Differentiation and the Derivative

The study of calculus usually begins with the basic definition of a *derivative*. A derivative is obtained through the process of *differentiation,* and the study of all forms of differentiation is collectively referred to as *differential calculus*.If we begin with a function and determine its derivative, we arrive at a new function called the *first derivative*. If we differentiate the *first derivative*, we arrive at a new function called the *second derivative*, and so on.

The derivative of a function is the *sl ope* at a given point.



Symbols for the Derivative

$$\frac{dy}{dx} = f'(x) = \frac{df(x)}{dx}$$

Definition $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Slope of a Piecewise Linear Segment

$$\frac{dy}{dx} = \text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

Development of a Simple Derivative

$$y = x^2$$

$$y + \Delta y = (x + \Delta x)^2$$

$$y + \Delta y = x^2 + 2x\Delta x + (\Delta x)^2$$

$$\Delta y = 2x\Delta x + (\Delta x)^2$$

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x$$

Chain Rule

$$y = f(u) \qquad u = t(x)$$

$$\frac{dy}{dx} = \frac{df(u)}{du}\frac{du}{dx} = f(u)\frac{du}{dx}$$

where $$f'(u) = \frac{df(u)}{du}$$

Example :Approximate the derivative of $y=x^2$ at $x=1$ by forming small changes.

$$y(1) = (1)^2 = 1$$

$$y(1.01) = (1.01)^2 = 1.0201$$

$$\Delta y = 1.0201 - 1 = 0.0201$$

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{0.0201}{0.01} = 2.01$$

Example : The derivative of sin u with respect to u is given below.

$$\frac{d}{du}(\sin u) = \cos u$$

Use the chain rule to find the derivative with r espect to *x* of

$$y = 4\sin x^2$$

Example:

$$u = x^2$$

$$\frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = f(u)\frac{du}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= 4(\cos u)(2x) = 8x\cos x^2$$

Some Derivatives

$f(x)$	$f'(x)$	Derivative Number
$af(x)$	$af'(x)$	D-1
$u(x)+v(x)$	$u'(x)+v'(x)$	D-2
$f(u)$	$f'(u)\frac{du}{dx}=\frac{df(u)}{du}\frac{du}{dx}$	D-3
a	0	D-4
$x^n \quad (n\neq 0)$	nx^{n-1}	D-5
$u^n \quad (n\neq 0)$	$nu^{n-1}\frac{du}{dx}$	D-6
uv	$u\frac{dv}{dx}+v\frac{du}{dx}$	D-7
$\frac{u}{v}$	$\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$	D-8
e^u	$e^u\frac{du}{dx}$	D-9

Some Derivatives

a^u	$(\ln a)a^u \dfrac{du}{dx}$	D-10
$\ln u$	$\dfrac{1}{u}\dfrac{du}{dx}$	D-11
$\log_a u$	$(\log_a e)\dfrac{1}{u}\dfrac{du}{dx}$	D-12
$\sin u$ $\cos u$	$\cos u\left(\dfrac{du}{dx}\right)$	D-13
	$-\sin u \dfrac{du}{dx}$	D-14
$\tan u$	$\sec^2 u \dfrac{du}{dx}$	D-15
$\sin^{-1} u$	$\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$ $\left(-\dfrac{\pi}{2} \le \sin^{-1} u \le \dfrac{\pi}{2}\right)$	D-16
$\cos^{-1} u$	$\dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx}$ $(0 \le \cos^{-1} u \le \pi)$	D-17
$\tan^{-1} u$	$\dfrac{1}{1+u^2}\dfrac{du}{dx}$ $\left(-\dfrac{\pi}{2} < \tan^{-1} u < \dfrac{\pi}{2}\right)$	D-18

Example: Determine *dy/dx* for the function shown below.

$$y = x^2 \sin x$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= x^2 \frac{d(\sin x)}{dx} + \sin x \frac{d(x^2)}{dx}$$

$$\frac{dy}{dx} = x^2 \cos x + \sin x (2x)$$

$$= x^2 \cos x + 2x \sin x$$

Example:. Determine *dy/dx* for the function shown below.

$$y = \frac{\sin x}{x}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{x\frac{d(\sin x)}{dx} - \sin x\frac{d(x)}{dx}}{x^2}$$

$$= \frac{x\cos x - \sin x}{x^2}$$

Example :Determine *dy/dx* for the function shown below.

$$y = e^{-\frac{x^2}{2}} \qquad u = -\frac{x^2}{2}$$

$$\frac{du}{dx} = \frac{d\left(-\frac{x^2}{2}\right)}{dx} = \left(-\frac{1}{2}\right)(2x) \Rightarrow$$

$$\frac{dy}{dx} = e^{-\frac{x^2}{2}}(-x) = -xe^{-\frac{x^2}{2}}$$

Higher-Order Derivatives

$$y = f(x)$$

$$\frac{dy}{dx} = f'(x) = \frac{df(x)}{dx}$$

$$\frac{d^2y}{dx^2} = f''(x) = \frac{d^2f(x)}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{d^3y}{dx^3} = f'''(x) = \frac{d^3f(x)}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

Example :Determine the 2nd derivative with respect to *x* of the function below.

$$y = \sin 4x$$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin 4x) = 4\cos 4x$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(4\cos 4x) = 4(-\sin 4x)(4) = -16\sin 4x$$

Applications: Maxima and Minima

1. Determine the derivative.
2. Set the derivative to 0 and solve for values that satisfy the equation.
3. Determine the second derivative.
   (a) If second derivative > 0, point is a *minimum*.
   (b) If second derivative < 0, point is a *maximum*.

Displacement, Velocity, and Acceleration

Displacement y

Velocity $$v = \frac{dy}{dt}$$

Acceleration $$a = \frac{dv}{dt} = \frac{d^2y}{dt^2}$$

Example:Determine local maxima or minima of function below.

$$y = f(x) = x^3 - 6x^2 + 9x + 1$$

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$

$$3x^2 - 12x + 9 = 0$$

$$x = 1 \text{ and } x = 3$$

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$

$$\frac{d^2y}{dx^2} = 6x - 12$$

For x = 1, $f''(1)$ = -6. Point is a *maximum* and y_{max} = 6.

For x = 3, $f''(3)$ = 6. Point is a *minimum* and y_{min} = 2.