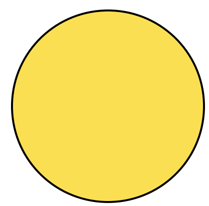
Phase Velocity and Group Velocity

Particle

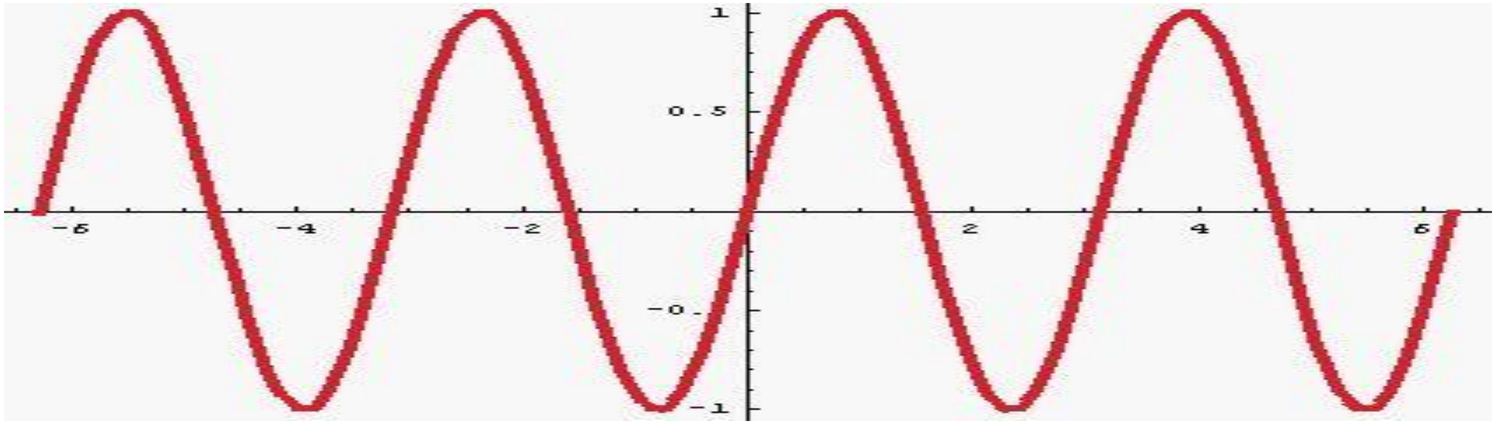
Our traditional understanding of a particle…



"Localized" - definite position, momentum, confined in space

Wave

Our traditional understanding of a wave….



"de-localized" – spread out in space and time

Waves

Normal Waves

- are a disturbance in space
- carry energy from one place to another
- often (but not always) will (approximately) obey the classical wave equation

Matter Waves

- disturbance is the wave function $\Psi(x, y, z, t)$
  - probability amplitude Ψ
  - probability density $p(x, y, z, t) = |\Psi|^2$

General Wave properties

Oscillations at a particlar point

$$y = A\cos 2\pi\nu t$$

travelling waves $x = V_p t$

$$y = A\cos 2\pi\nu(t - x/V_p)$$

$$y = A\cos 2\pi(\nu t - \nu x/V_p)$$

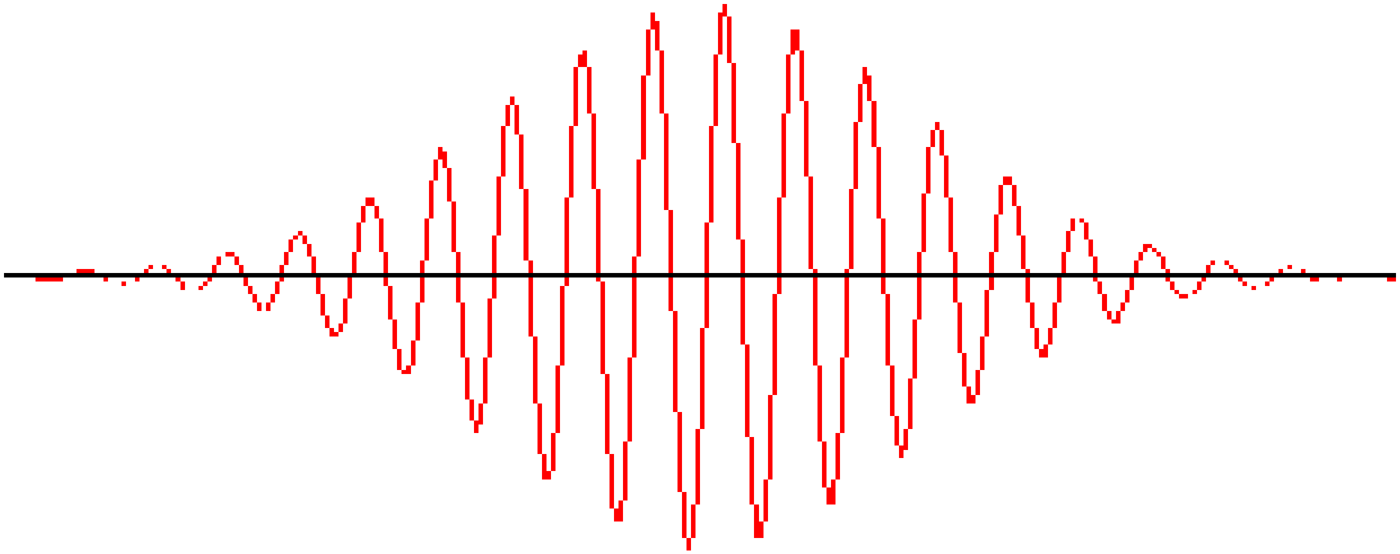
$but V_p = \nu\lambda$

$$y = A\cos(\omega t - kx)$$

$$\omega = 2\pi\nu \text{ (angular frequency)}$$

$$k = \frac{2\pi}{\lambda} \text{ (wave number)}$$

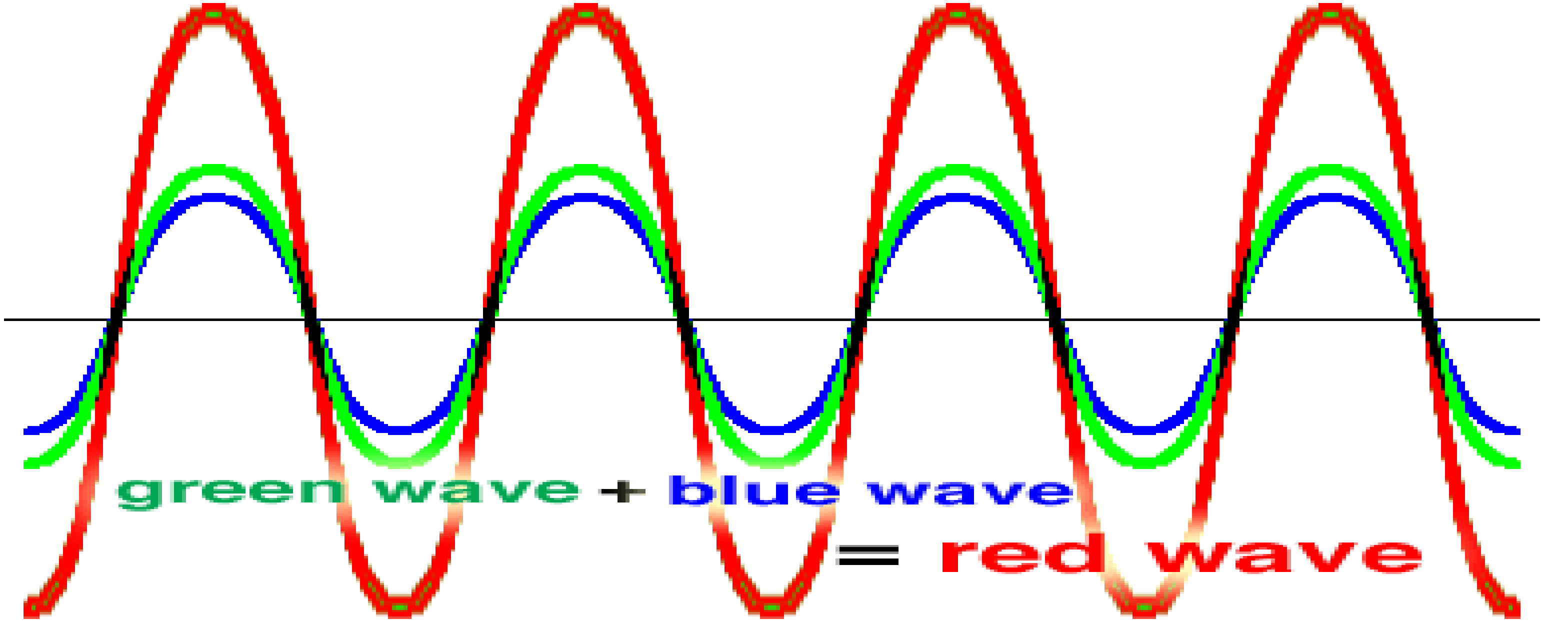
A "Wave Packet"



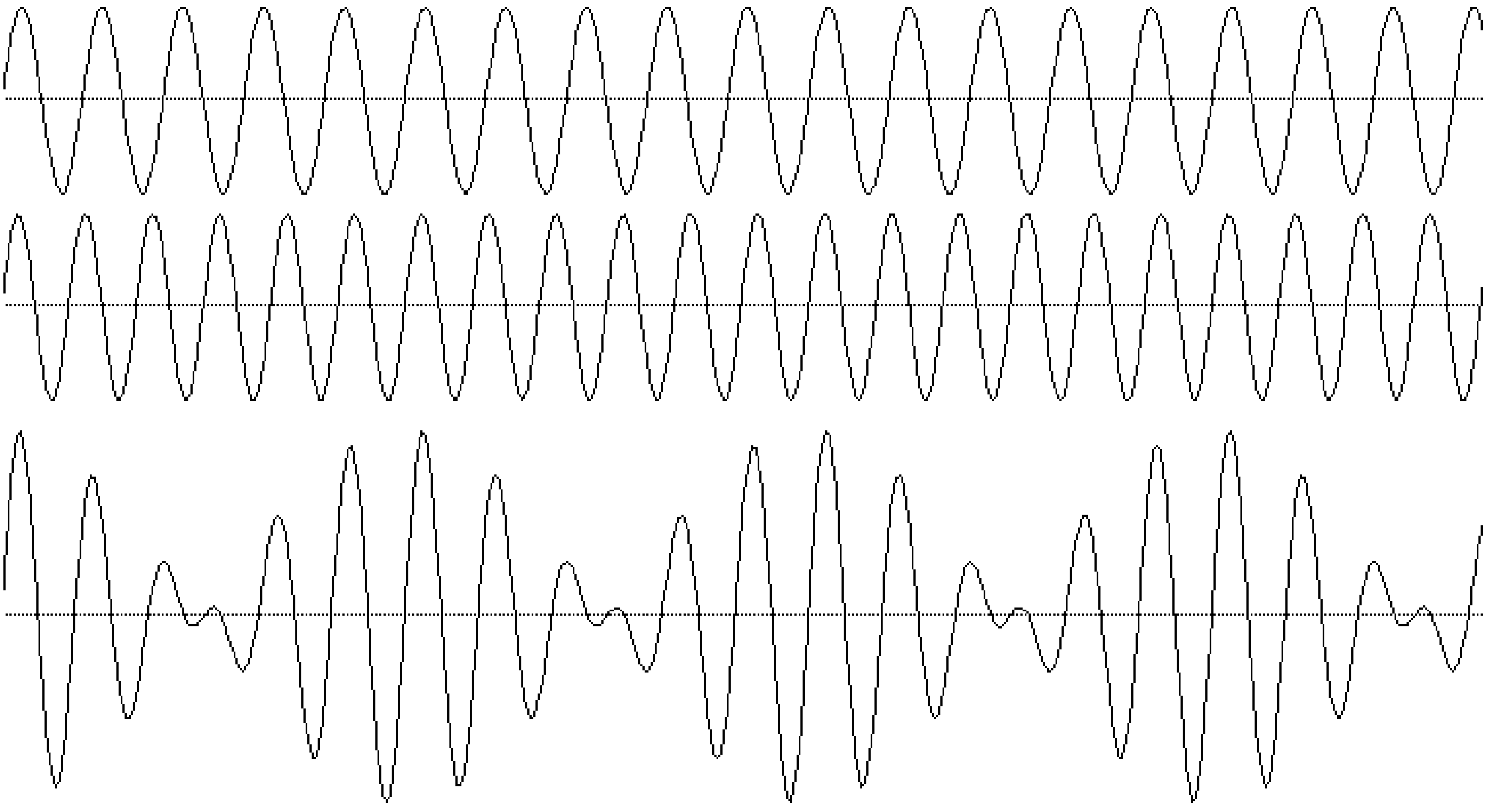
How do you construct a wave packet?

What happens when you add up waves?

The Superposition principle



Adding up waves of different frequencies.....



Constructing a wave packet by adding up several waves ………..

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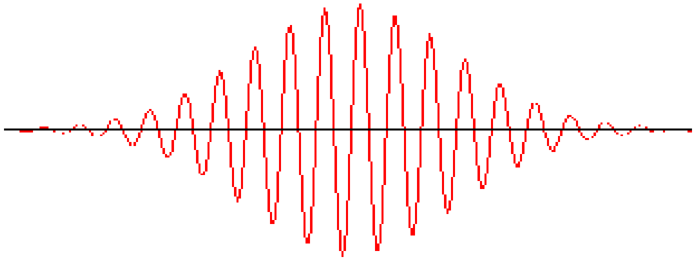
If several waves of different wavelengths (frequencies) and phases are superposed together, one would get a resultant which is a localized wave packet



A wave packet describes a particle

- A wave packet is a group of waves with slightly different wavelengths interfering with one another in a way that the amplitude of the group (envelope) is non-zero only in the neighbourhood of the particle

- A wave packet is localized – a good representation for a particle!

Wave packet, phase velocity and group velocity

- The spread of wave packet in wavelength depends on the required degree of localization in space – the central wavelength is given by

$$\lambda = \frac{h}{p}$$

- What is the velocity of the wave packet?

Wave packet, phase velocity and group velocity

- The velocities of the individual waves which superpose to produce the wave packet representing the particle are different - the wave packet as a whole has a different velocity from the waves that comprise it
- Phase velocity: The rate at which the phase of the wave propagates in space
- Group velocity: The rate at which the envelope of the wave packet propagates

Phase velocity

- Phase velocity is the rate at which the phase of the wave propagates in space.
- This is the velocity at which the phase of any one frequency component of the wave will propagate.
- You could pick one particular phase of the wave and it would appear to travel at the phase velocity.
- The phase velocity is given in terms of the wave's angular frequency ω and wave vector *k* by

$$v_{\mathrm{p}} = \frac{\omega}{k}$$

Group velocity

Group velocity of a wave is the velocity with which the variations in the shape of the wave's amplitude (known as the **modulation** or **envelope** of the wave) propagate through space.
The group velocity is defined by the equation

$$v_g \equiv \frac{\partial \omega}{\partial k},$$

where:

vg is the group velocity;
ω is the wave's angular frequency;
k is the wave number.

The function *ω*(*k*), which gives *ω* as a function of *k*, is known as the dispersion relation.

Velocity of De-Broglie Waves-Phase velocity

- Moving material particle-De broglie said a wave is associated which travel with the same velocity as that of the wave.

- Let De-broglie wave velocity is V_p, so

$$V_p = \upsilon\lambda$$

Phase Velocity - continued

- we know that

$$\lambda = h/mv$$

- From Planck`s Law

$$E = h\nu$$

- From mass energy relation

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$$E = mc^2$$

Phase Velocity - continued

- $h\nu = mc^2$ $\nu = mc^2/h$

  $Vp = \upsilon \lambda$

Therefore

$$Vp = mc^2/h \text{ X } h/mv$$

- $Vp = C^2/v$
- So Particle velocity must be less than the velocity of light. $Vp > c$
- We can say that De-Broglie waves must travel faster than the velocity of light.

Group Velocity=Particle velocity

$$y_1 = A\cos(\omega t - kx)$$

$$y_2 = A\cos[(\omega + d\omega)t - (k + dk)x]$$

$$y = y_1 + y_2$$

$$y = A\cos(\omega t - kx) + A\cos[(\omega + d\omega)t - (k +$$

$$y = 2A\cos\left[\frac{(2\omega + d\omega)t}{2} + \frac{(2k + dk)x}{2}\right]\cos\left[\frac{(d\omega)t}{2} - \frac{(dk)x}{2}\right]$$

$$with \quad d\omega << \omega, dk \ \ << k$$

$$y \cong 2A\cos\left[\frac{d\omega}{2}t - \frac{dk}{2}x\right]\cos[\omega t - kx] ------ -(1)$$

Group Velocity=Particle velocity

- Equation1 represent a wave of angular velocity w and wave number k which has superimposed upon it a wave (the process is called modulation) of angular velocity $\frac{d\omega}{2}$ and wave number $\frac{dk}{2}$

phase velocity= wave velocity of carrier : $v_p = \frac{\omega}{k}$

group velocity= wave velocity of envelope : $v_g = \frac{\Delta\omega}{\Delta k}$

for more than two wave contiributions: $v_g = \frac{d\omega}{dk}$

De Broglie waves for massive particles

$$\omega = 2\pi\nu$$

$$\omega = 2\pi \frac{E}{h}$$

$$\omega = 2\pi \frac{mc^2}{h}$$

$$\boxed{\omega = 2\pi \frac{m_0 c^2}{h\sqrt{1 - \frac{v^2}{c^2}}}}$$

$$v_g = \frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv}$$

$$\omega = \frac{2\pi}{h}\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$d\omega/dv = \frac{2\pi m_0 c^2}{h}(-\frac{1}{2})\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left[-2\,v/c^2\right]$$

$$d\omega/dv = \frac{2\pi}{h}\frac{m_0 v}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}$$

$$k = \frac{2\pi}{h}\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$dk/dv = \frac{2\pi m_0}{h}\left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} + (-\frac{1}{2})\left(1-(v/c)^2\right)^{-3/2}\left[-2v/c^2\right]\right]$$

$$dk/dv = \frac{2\pi m_0}{h}\left(1-(v/c)^2\right)^{-3/2}\left[1-\frac{v^2}{c^2}+\frac{v^2}{c^2}\right]$$

$$dk/dv = \frac{2\pi m_0}{h}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}$$

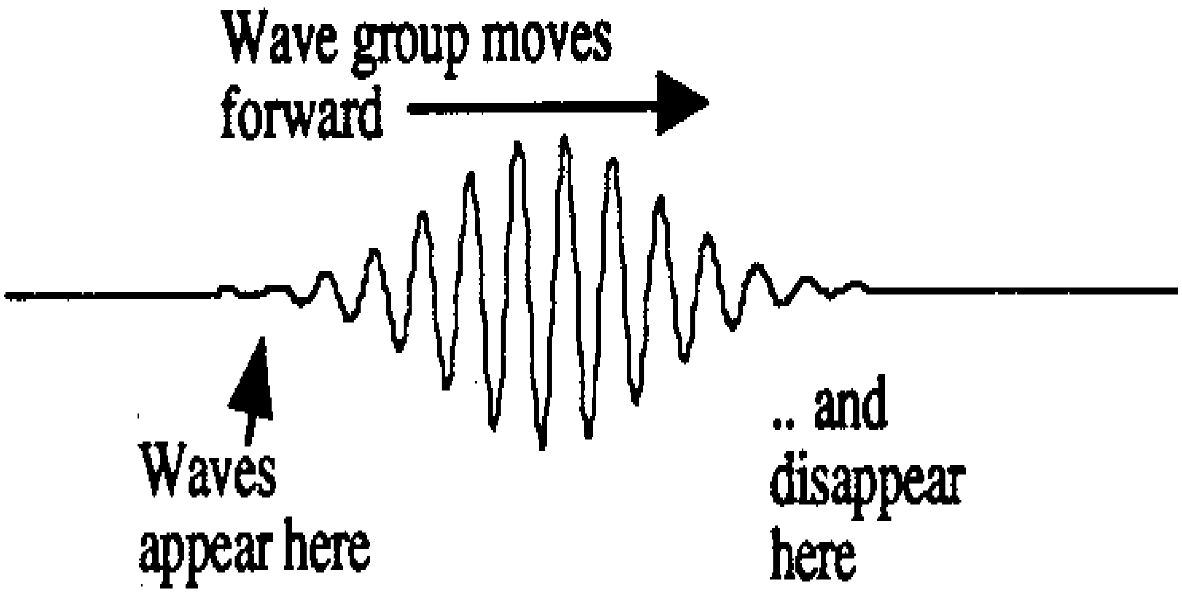
$$v_g = \frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv}$$

$$d\omega/dv = \frac{2\pi}{h}\frac{m_0 v}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}$$

$$dk/dv = \frac{2\pi m_0}{h}\frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}$$

$$\Rightarrow v_g = v$$

Scheme



Wave group moves forward
Waves appear here
.. and disappear here